

# Defining the space in a general spacetime

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## Abstract

A global vector field  $v$  on a “spacetime” differentiable manifold  $V$ , of dimension  $N + 1$ , defines a congruence of world lines: the maximal integral curves of  $v$ , or orbits. The associated global space  $N_v$  is the set of these orbits. A “ $v$ -adapted” chart on  $V$  is one for which the  $\mathbb{R}^N$  vector  $\mathbf{x} \equiv (x^j)$  ( $j = 1, \dots, N$ ) of the “spatial” coordinates remains constant on any orbit  $l$ . We consider non-vanishing vector fields  $v$  that have non-periodic orbits, each of which is a closed set. We prove transversality theorems relevant to such vector fields. Due to these results, it can be considered plausible that, for such a vector field, there exists in the neighborhood of any point  $X \in V$  a chart  $\chi$  that is  $v$ -adapted and “nice”, i.e., such that the mapping  $\bar{\chi} : l \mapsto \mathbf{x}$  is injective — unless  $v$  has some “pathological” character. This leads us to define a notion of “normal” vector field. For any such vector field, the mappings  $\bar{\chi}$  build an atlas of charts, thus providing  $N_v$  with a canonical structure of differentiable manifold (when the topology defined on  $N_v$  is Hausdorff, for which we give a sufficient condition met in important physical situations). Previously, a local space manifold  $M_F$  had been associated with any “reference frame”  $F$ , defined as an equivalence class of charts. We show that, if  $F$  is made of nice  $v$ -adapted charts,  $M_F$  is naturally identified with an open subset of the global space manifold  $N_v$ .

*Keywords:* Physical space; global vector field; reference fluid; orbit space; adapted chart; differentiable manifold; Kruskal-Szekeres coordinates.

# 1 Introduction

## 1.1 Physical motivation

The theory of relativity says that space and time merge into “a kind of union of the two” (in Minkowski’s words): the spacetime. However, the notion of a *physical space* should be useful also in relativistic physics. In our opinion it is even needed, for the following two reasons. (i) In experimental/observational work, one of course needs to define the spatial position of the experimental apparatus and/or of the observed system, and this is true also if the relativistic effects have to be considered. (ii) In quantum mechanics, the “state” of a quantum-mechanical particle is a function  $\psi$  of the position  $x$  belonging to some 3-D “physical space”  $M$ , and taking values in  $\mathbb{C}$  (or in a complex vector bundle). Note that defining such a space as a spacelike 3-D *submanifold of the spacetime manifold* (e.g. [1]) can work to define an initial condition for a field in space-time, but does not allow one to define a spatial position in the way that is needed in the two foregoing examples: in those, one needs to identify spatial points that exist at least for some open interval of time — e.g. to state that some objects maintain a fixed spatial position in some reference frame, or to define the stationary states of a quantum particle. In practice, the spatial position is taken to be the triplet of the spatial coordinates,  $\mathbf{x} \equiv (x^j)$  ( $j = 1, 2, 3$ ). However, *a priori*,  $\mathbf{x}$  does not have a precise geometric meaning in a theory starting from a spacetime structure. Only a notion of “spatial tensors” has been defined for a general spacetime of relativistic gravity. This definition was based on the concept of “reference fluid” [2, 3, 4], also named “reference body” [5] — i.e., a three-dimensional congruence of world lines, whose the tangent vector is assumed to be a time-like vector field. The latter can be normed to become a four-velocity field  $v$ , i.e.  $\mathbf{g}(v, v) = 1$  where  $\mathbf{g}$  is the spacetime metric. The data of the four-velocity field  $v$  allows one to define the spatial projection operator  $\Pi_X$  (depending on the point  $X$  in the spacetime manifold  $V$ ) [2, 3, 4, 5, 6, 7]. A “spatial vector” is then defined as a *spacetime vector* which is equal to its spatial projection. A full algebra of “spatial tensors” can be defined in the same way, and also, once a relevant connection has been defined, a spatial tensor analysis [2, 3, 4, 5, 6].

However, it is possible in a general spacetime manifold  $V$  to define a relevant physical space as a *3-D differentiable manifold*, at least *locally* in  $V$ .

To see this, consider a coordinate system or chart:

$$\chi : U \rightarrow \mathbb{R}^4, \quad X \mapsto \chi(X) = \mathbf{X} \equiv (x^\mu) \quad (\mu = 0, \dots, 3), \quad (1)$$

where  $U$  is an open subset of  $V$ : the domain of the chart. Then one may define a set of world lines, each of which,  $l$ , has constant spatial coordinates  $a^j$  in the chart  $\chi$ :<sup>1</sup>

$$l_{\mathbf{a}} = \{X \in U; \chi(X) = (x^\mu) \text{ is such that } x^j = a^j \text{ for } j = 1, 2, 3\}. \quad (3)$$

Let us suppose for a moment that the chart  $\chi$  is in fact a Cartesian coordinate system on the Minkowski spacetime. Then that chart defines an inertial reference frame. In that case, it is clear that, for any event  $X$ , with  $\chi(X) = (x^\mu)$ , the triplet  $\mathbf{x} \equiv (x^j)$  ( $j = 1, 2, 3$ ) defines the spatial position associated in that chart with the event  $X$ . Note that the data of  $\mathbf{x}$  is equivalent to specifying a unique world line in the “congruence (3)”. [By this we mean the set of the world lines (3), when  $\mathbf{a} \equiv (a^j)$  takes any value in  $\mathbb{R}^3$  such that the corresponding world line (3) is not empty.] That world line is thus uniquely determined by the event  $X$  and may be noted  $l(X)$ . Events  $X'$  that have different values of the time coordinate  $x^0$ , but that have the same values of the spatial coordinates  $x^j$  ( $j = 1, 2, 3$ ), can be said to occur at the same spatial position in the inertial frame as does  $X$ . Thus, the whole of  $l(X)$  is needed. However, each world line in the congruence (3) stays invariant if we change the coordinate system by a purely spatial coordinate change:

$$x'^0 = x^0, \quad x'^k = \phi^k((x^j)). \quad (4)$$

It is clear that this transformation leaves us in the same inertial reference frame. With the new chart  $\chi'$ , the new triplet  $\mathbf{x}' \equiv (x'^k) \equiv \phi(\mathbf{x})$  corresponds to the same spatial position in the inertial frame as does  $\mathbf{x}$  with the first chart  $\chi$ . And indeed, that world line of the congruence which is defined in the chart  $\chi'$  by the data of  $\mathbf{x}'$  is just the same as that world line of the congruence which is defined in the chart  $\chi$  by the data of  $\mathbf{x}$ . The spatial position

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<sup>1</sup> Note that, if we assume that  $V$  is endowed with a Lorentzian metric  $\mathbf{g}$  whose component  $g_{00}$  in the chart  $\chi$  verifies  $g_{00} > 0$  in  $U$ , then each among the world lines  $l$  is time-like, because in the chart  $\chi$  the tangent vector to  $l$  has components  $\propto (1, 0, 0, 0)$ , which may be normed to

$$v^0 \equiv \frac{1}{\sqrt{g_{00}}}, \quad v^j = 0. \quad (2)$$

of the event  $X$  in the inertial frame is therefore most precisely defined by *the world line  $l(X)$  of the congruence which passes at  $X$* .

Now note that very little in the foregoing paragraph actually depends on whether or not the chart  $\chi$  is a Cartesian coordinate system on the Minkowski spacetime: only the qualification of the reference frame as being inertial depends on that. It is just that we are accustomed to consider a spatial position in an inertial frame in a flat spacetime, and special relativity makes it natural to accept that it is actually the world line  $l(X)$  which best represents that spatial position. Hence, consider a general spacetime, and define a congruence of world lines from the data of a coordinate system as in Eq. (3). In the domain of the chart, we may then define the spatial position of an event  $X$  as the unique world line  $l(X)$  of the congruence (3) which passes at  $X$  — i.e.,  $l(X)$  is the unique world line of the congruence (3), such that  $X \in l(X)$ . Thus, *the data of a coordinate system on the spacetime defines a three-dimensional space  $M$ , of which the points (the elements of  $M$ ) are the world lines of the congruence (3) associated with that coordinate system*.

## 1.2 Local reference frame and local space manifold

The foregoing approach can be used to define precise notions of a reference frame and its unique associated space manifold [8]. First, the invariance of the congruence (3) under the purely spatial coordinate changes (4) allows one to define a *reference frame* as being an equivalence class of charts related by a change (4). More exactly, the following is an equivalence relation between charts which are defined on a given open subspace  $U$  of the spacetime manifold  $V$ :

$$\chi \mathcal{R}_U \chi' \iff [\forall \mathbf{X} \in \chi(U), \quad \phi^0(\mathbf{X}) = x^0 \text{ and } \frac{\partial \phi^k}{\partial x^0}(\mathbf{X}) = 0 \ (k = 1, \dots, 3)], \quad (5)$$

where  $f \equiv \chi' \circ \chi^{-1} \equiv (\phi^\mu)$  is the transition map, which is defined on  $\chi(U)$ . Thus a reference frame  $F$  is a set of charts defined on the same open domain  $U$  and exchanging by a purely spatial coordinate change (4). Then the *space manifold*  $M$  or  $M_F$  associated with the reference frame  $F$  is defined as the set of the world lines (3). In detail: let  $P_S : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ ,  $\mathbf{X} \equiv (x^\mu) \mapsto \mathbf{x} \equiv (x^j)$ , be the spatial projection. A world line  $l$  is an element of  $M_F$  iff there is a chart  $\chi \in F$  and a triplet  $\mathbf{x} \in P_S(\chi(U))$ , such that  $l$  is the set of *all* points  $X$  in

the domain  $U$ , whose spatial coordinates are  $\mathbf{x}$ :

$$l \equiv \{ X \in U; P_S(\chi(X)) = \mathbf{x} \}. \quad (6)$$

(Thus,  $l$  is not necessarily a connected set.) It results easily from (4) that (6) holds true then in any chart  $\chi' \in F$ , of course with the transformed spatial projection triplet  $\mathbf{x}' = \phi(\mathbf{x}) \equiv (\phi^j(\mathbf{x}))$  [8]. For a chart  $\chi \in F$ , one defines the “associated chart” as the mapping which associates, with a world line  $l \in M$ , the constant triplet of the spatial coordinates of the points  $X \in l$ :

$$\tilde{\chi} : M \rightarrow \mathbb{R}^3, \quad l \mapsto \mathbf{x} \text{ such that } \forall X \in l, P_S(\chi(X)) = \mathbf{x}. \quad (7)$$

One shows then that the set  $\mathcal{T}$  of the subsets  $\Omega \subset M$  such that,

$$\forall \chi \in F, \quad \tilde{\chi}(\Omega) \text{ is an open set in } \mathbb{R}^3 \quad (8)$$

is a topology on  $M$ . Finally one shows that the set of the associated charts:  $\tilde{F} \equiv \{\tilde{\chi}; \chi \in F\}$ , is an atlas on the topological space  $(M, \mathcal{T})$ , hence defines a structure of *differentiable manifold* on  $M$  [8]. *Thus the space manifold  $M_F$  is browsed by precisely the triplet  $\mathbf{x} \equiv (x^j)$  made with the spatial projection of the spacetime coordinates  $\mathbf{X} \equiv (x^\mu) \equiv \chi(X)$  of a chart  $\chi \in F$ , see Eq. (7).*

Using these results, one may define the space of quantum-mechanical states, for a given reference frame  $F$  in a given spacetime  $(V, \mathbf{g})$ , as being the set  $\mathcal{H}$  of the square-integrable functions defined on the corresponding space manifold  $M$  [9]. One may also define the full algebra of spatial tensors: the pointwise algebra is defined simply as the tensor algebra of the tangent vector space  $TM_x$  to the space manifold  $M$  at some arbitrary point  $x \in M$  [10].

### 1.3 Goal and summary

Thus, by defining a reference frame as a set  $F$  of charts that all have the same open domain  $U$  and that exchange by a purely spatial coordinate change (4), one can then define the associated space manifold  $M_F$  as the set of the world lines (6) [8]. These definitions are relevant to physical applications [9, 10]. However, they apply to a *parametrizable* domain  $U$  of the spacetime manifold  $V$ , i.e., to an open set  $U$ , such that at least one regular chart can be defined over the whole of  $U$ . Since the manifold  $V$  itself as a whole is in general not

parametrizable, a reference frame is in general only a local one, and so the associated space manifold does not look “maximal”. *The aim of the present work* is to define *global* reference fluids, to associate with any of them a *global physical space*, and to *link* these concepts with the formerly defined *local concepts*. As the “spacetime”, we consider a differentiable manifold  $V$  having dimension  $N + 1$ , thus  $N$  is the dimension of the “space” manifold to be defined. We define a global reference fluid by the data of a *non-vanishing* global vector field  $v$  on  $V$ . We do not need that  $N = 3$ , nor that  $V$  be endowed with a Lorentzian metric  $\mathbf{g}$  for which  $v$  be a time-like vector field. This was already true for the former “local” work [8]. Note, however, that a time-like vector field on a Lorentzian manifold  $(V, \mathbf{g})$  is non-vanishing; and that, if a Lorentzian manifold  $(V, \mathbf{g})$  is time-oriented, which indeed is usually required for a spacetime, then by definition there exists at least one global time-like vector field on  $V$ . We define the “global space” associated with  $v$  as the set  $N_v$  of the maximal integral curves (or “orbits”) of  $v$ . To reach our goal, we take the following steps:

(a) Section 2 studies when a given vector field  $v$  on a differentiable manifold  $V$  is such that there locally exists charts of  $V$  which are *adapted* [2] to the congruence associated with  $v$ ; i.e., charts in which the “spatial” position  $\mathbf{x} \equiv (x^j)$  ( $j = 1, \dots, N$ ) is constant on any orbit  $l$  of  $v$ ; see Definition 1. We need also that the mapping  $l \mapsto \mathbf{x}$  be injective. The desired situation is defined by Proposition 2. According to a transversality argument, this situation should be attainable, in general, if  $v$  does not vanish and each of its orbits is non-periodic and is closed in  $V$ . In Subsect. 2.4, two theorems of transversality and another theorem pertaining to differential topology allow us to formalize that argument in Theorem 4. This justifies us in introducing a notion of “normal” vector field by Definition 2, which ensures the local existence of  $v$ -adapted charts through Theorem 5.

(b) For any  $v$ -adapted chart  $\chi$ , one may define the mapping  $\bar{\chi}$  which associates with any orbit  $l$  the constant spatial position  $\mathbf{x}$ . We show in Section 3 that, using the set  $\mathcal{A}$  of the injective mappings  $\bar{\chi}$ , one can endow the global orbit set  $N_v$  with a topology  $\mathcal{T}'$  for which this set is an atlas of charts. See Theorem 6. Thus, when  $\mathcal{T}'$  is metrizable and separable, we do have a canonical structure of differentiable manifold on the orbit set  $N_v$ , for which the mappings  $\bar{\chi}$  defined by Eq. (14) are local charts on the “space” manifold  $N_v$ . I.e., if  $N = 3$ , *also the global space  $N_v$  is browsed (locally) by precisely the*

triplet  $\mathbf{x} \equiv (x^j)$  made with the spatial projection of the spacetime coordinates  $\mathbf{X} \equiv (x^\mu) \equiv \chi(X)$  of a  $v$ -adapted chart  $\chi$ .

(c) In Section 4 we establish the link with the previously defined space manifold  $M_F$ , associated with a given local reference frame  $F$  — defined as an equivalence class of charts for the relation (5). We show in Theorem 7 that, when the charts belonging to  $F$  are  $v$ -adapted and with the mapping  $l \mapsto \mathbf{x}$  being injective, then  $M_F$  is naturally identified with an open subset of  $N_v$ .

The definitions of a local reference frame  $F$  and the corresponding “space” manifold  $M_F$  do not need any metrical structure on the “spacetime” manifold  $V$  [8]. Just the same can be said for the definition of the global “space” manifold  $N_v$ , beyond the very fact that  $V$  should have a metrizable topology.

## 2 Local existence of adapted charts

### 2.1 Definitions

Let  $V$  be an  $(N + 1)$ –dimensional real differentiable manifold, with  $N \geq 1$ .<sup>2</sup> We consider a given global, smooth vector field  $v$  on  $V$ . A continuously differentiable ( $\mathcal{C}^1$ ) mapping  $C$  from an open interval  $I$  of  $\mathbb{R}$  into  $V$  defines an integral curve of  $v$  iff  $\frac{dC}{ds} = v(C(s))$  for  $s \in I$ . For any  $X \in V$ , let  $C_X$  be the solution of

$$\frac{dC}{ds} = v(C(s)), \quad C(0) = X \quad (9)$$

for which the open interval  $I$  is *maximal*, and denote this maximal interval by  $I_X$  [12]. That is,  $I_X$  is an open interval defined as the union of all open intervals  $I$ , each containing 0, in which a solution of (9) is defined. The solution  $C_X$  is defined on  $I_X$  and is unique [12]. Let  $s \in I_X$  and set  $Y = C_X(s)$ . These definitions imply easily [12] that

$$I_Y = I_X - s \quad \text{and} \quad \forall t \in I_Y, \quad C_Y(t) = C_X(s + t). \quad (10)$$

For any  $X \in V$ , call the *range*  $l_X \equiv C_X(I_X) \subset V$  the “maximal integral curve at  $X$ ”. From (10), “ $l_X$  does not depend on the point  $X \in l_X$ ”: if  $Y \in l_X$ ,

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<sup>2</sup> We understand “differentiable manifold” as a topological space endowed with an atlas of compatible charts, hence with the corresponding equivalence class of compatible atlases — with the restriction that that space should be metrizable and separable [11].

then  $l_Y = l_X$ . We define the set of the maximal integral curves (or orbits) of  $v$ :

$$N_v \equiv \{l_X; X \in V\}. \quad (11)$$

Once endowed with further structure,  $N_v$  will be the global space manifold associated with the global vector field  $v$  (when the latter is non-vanishing and obeys another assumption). Note that, if the set  $U \subset V$  is not empty, then the following subset of  $N_v$ :

$$D_U \equiv \{l \in N_v; l \cap U \neq \emptyset\} \quad (12)$$

is not empty. Indeed, for any  $X \in U$ , the world line  $l \equiv l_X$  belongs to  $D_U$ . Let  $P_S : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N$ ,  $(x^\mu) \mapsto (x^j)$  ( $\mu = 0, \dots, N$ ;  $j = 1, \dots, N$ ) be the “spatial” projection.

**Definition 1.** *A mapping  $\chi : U \rightarrow \mathbb{R}^{N+1}$  with  $U \subset V$  is said “ $v$ -adapted” iff for any  $l \in D_U$ , there exists  $\mathbf{x} \in \mathbb{R}^N$  such that*

$$\forall Y \in l \cap U, \quad P_S(\chi(Y)) = \mathbf{x}. \quad (13)$$

If Eq. (13) is verified by some world line  $l \in N_v$ , then necessarily  $l \in D_U$ , and  $\mathbf{x}$  is obviously unique. Thus, for any  $v$ -adapted mapping  $\chi$ , the mapping

$$\bar{\chi} : D_U \rightarrow \mathbb{R}^N, \quad l \mapsto \mathbf{x} \text{ such that (13) is verified} \quad (14)$$

is well defined. In Section 3, we will endow the set  $N_v$  with first a topology and then a structure of differentiable manifold, for which the charts (of  $N_v$ ) will be mappings  $\bar{\chi}$ , where  $\chi$  is a  $v$ -adapted *chart* of  $V$ . Since any chart is in particular a one-to-one mapping, we need to restrict ourselves to  $v$ -adapted charts  $\chi$  such that the associated mapping  $\bar{\chi}$  is injective. Thus, we define that a  $v$ -adapted chart  $\chi$  is “*nice*” iff  $\bar{\chi}$  is injective on  $D_U$ , with  $U \subset V$  the (open) domain of  $\chi$ .

## 2.2 Straightening-out vs $v$ -adapted charts

In the remainder of this section, we investigate whether there exist nice  $v$ -adapted charts in the neighborhood of any point  $X_0 \in V$ . If the vector field  $v$  does not vanish, a well-known theorem applies at any point  $X_0 \in V$ :



**Straightening-out theorem** (e.g. [13]). *Let  $v$  be a vector field of class  $C^\infty$  defined on  $V$ . Suppose that at  $X_0 \in V$  we have  $v(X_0) \neq 0$ . There is a “straightening-out chart”  $\chi$  defined on an open neighborhood  $U$  of  $X_0$ , i.e.  $\chi$  is such that:*

- (i)  $\chi(U) = I \times \Omega$ ,  $I = ]-a, +a[$ ,  $a \neq 0$ ,  $\Omega$  open set in  $\mathbb{R}^N$ .
- (ii) For any  $\mathbf{x} \in \Omega$ ,  $\chi^{-1}(I \times \{\mathbf{x}\})$  is an integral curve of  $v$ .
- (iii) In  $U$ , we have  $v = \partial_0$ , where  $(\partial_\mu)$  is the natural basis associated with the chart  $\chi$ .

However, the direct link with the notion of a  $v$ -adapted chart works in the wrong direction:

**Proposition 0.** *Let  $(\chi, U)$  be a  $v$ -adapted chart. (i) We have  $v|_U = f\partial_0$  with  $f : U \rightarrow \mathbb{R}$  a smooth function. (ii) Given any point  $X \in U$  such that  $v(X) \neq 0$ , one may obtain a straightening-out chart  $(\chi', U')$ , with  $U' \subset U$  being an open neighborhood of  $X$ , by changing merely the  $y^0$  coordinate.*

*Proof.* (i) To say that  $(\chi, U)$  is  $v$ -adapted means, according to Definition 1, that for any given  $X \in U$ , we have for any  $Y \in l_X \cap U$ :  $P_S(\chi(Y)) = P_S(\chi(X))$ . In the coordinates  $\chi(X) = (x^\mu) \equiv \mathbf{X}$ ,  $\chi(Y) = (y^\mu) \equiv \mathbf{Y}$  ( $\mu = 0, \dots, N$ ), the latter rewrites as

$$y^j = x^j \quad (j = 1, \dots, N). \quad (15)$$

On the other hand, remembering the definition of  $l_X$  as a (maximal) integral curve of  $v$ , Eq. (9) and below, let  $J$  be the connected component of 0 in  $I'_{XU} \equiv \{s \in I_X; C_X(s) \in U\}$ :  $J$  is an open interval containing 0 and we have  $l' \equiv C_X(J) \subset l_X \cap U$ . Let us denote this as  $l' = \{Y(s); s \in J\}$ , with

$$\frac{dy^\mu}{ds} = v^\mu(\mathbf{Y}(s)) \quad (\mu = 0, \dots, N), \quad \mathbf{Y}(0) = \mathbf{X}, \quad (16)$$

$v^\mu = v^\mu(\mathbf{Y})$  being the components of  $v$  in the chart  $\chi$ . It follows from (15) and (16) that for  $s \in J$  we have  $v^j(\mathbf{Y}(s)) = 0$  ( $j = 1, \dots, N$ ), i.e.  $v(Y(s)) = v^0(\mathbf{Y}(s))\partial_0(Y(s))$ , thus in particular  $v(X) = v^0(\mathbf{X})\partial_0(X) \equiv f(X)\partial_0(X)$ . Since this is true at any point  $X \in U$ , our statement (i) is proved.

(ii) If we leave the coordinates  $y^j$  ( $j = 1, \dots, N$ ) unchanged:  $\chi'(Y) \equiv \mathbf{Y}' = (g(\mathbf{Y}), (y^j))$  where  $\mathbf{Y} = (y^0, (y^j)) = \chi(Y)$  ( $Y \in U' \subset U$ ), then the components of  $v$  in the new chart  $(\chi', U')$  are  $v'^j = v^j = 0$  ( $j = 1, \dots, N$ ) and  $v'^0(\mathbf{Y}') = \frac{\partial g}{\partial y^0} v^0(\mathbf{Y}) = \frac{\partial g}{\partial y^0} f(Y)$ . The latter must be equal to 1 for a straightening-out chart. Since  $v|_U = f \partial_0$  and  $v(X) \neq 0$ , we have  $f(Y) \neq 0$  when  $Y$  is in some neighborhood  $U'' \subset U$  of  $X$ . Hence, we get  $v'^0 = 1$  in  $U'$  if we take  $U' = \chi^{-1}(B)$  with  $B = ]x^0 - r, x^0 + r[ \times \dots \times ]x^N - r, x^N + r[ \subset \chi(U'')$  and define  $y'^0 \equiv g(\mathbf{Y}') = \int_{x^0}^{y^0} du/f(u, (y^j))$  for  $\mathbf{Y}' \in B$ . Property (i) in the straightening-out theorem is then got by a mere shift  $y'^0 \mapsto y'^0 + \delta$ , and its Property (ii) is a straightforward consequence of Properties (i) and (iii).  $\square$

Conversely, if  $(\chi, U)$  is a straightening-out chart, i.e. it fulfills conditions (i) to (iii) of the theorem above, then let  $\mathbf{x} \in \Omega$  and set  $l' \equiv \chi^{-1}(I \times \{\mathbf{x}\})$ . From (ii),  $l'$  is an integral curve of  $v$  and, since  $l' \subset U$  by construction, we have (13):

$$\forall Y \in l' \cap U, \quad P_S(\chi(Y)) = \mathbf{x}. \quad (17)$$

Hence, at first sight, it might seem that  $\chi$  is a  $v$ -adapted chart. However, let  $X = \chi^{-1}(s, \mathbf{x}) \in l'$  and let  $l \equiv l_X \in N_v$  be the *maximal* integral curve of  $v$  passing at  $X$ . We have  $l' \subset (l \cap U)$  since  $l'$  is an integral curve of  $v$  that is included in  $U$  and that passes at  $X$ . But nothing guarantees that  $l' \supset (l \cap U)$ : the intersection  $l \cap U$  may contain other arcs, say  $l'_1 \equiv \chi^{-1}(I \times \{\mathbf{x}_1\})$  with  $\mathbf{x}_1 \in \Omega$  and  $\mathbf{x}_1 \neq \mathbf{x}$ . In such a case, the straightening-out chart  $\chi$  is not  $v$ -adapted, since for  $Y \in l \cap U$ ,  $P_S(\chi(Y))$  may take different values  $\mathbf{x}, \mathbf{x}_1, \dots$

As we already noted, Property (ii) in the straightening-out theorem follows easily from Properties (i) and (iii). More generally, if a chart  $(\chi, U)$  satisfies Property (iii), i.e.  $v = \partial_0$  in  $U$ , and if  $\chi(U)$  contains a set  $I \times \{\mathbf{x}\}$ , with  $\mathbf{x} \in \mathbb{R}^N$  and  $I$  an open interval, then Property (ii) applies for this  $\mathbf{x} \in \mathbb{R}^N$ . Later on, we will need that the boundary of the open set  $U$  be a smooth hypersurface, hence we must consider charts for which (iii) is true, but not (i).

**Proposition 1.** *Let  $(\chi, U)$  be a chart such that  $v = \partial_0$  in  $U$ . Assume there is an open subset  $\Omega \subset \mathbb{R}^N$  such that*

$$\chi(U) = \bigcup_{\mathbf{x} \in \Omega} I_{\mathbf{x}} \times \{\mathbf{x}\}, \quad (18)$$

where  $I_{\mathbf{x}}$  ( $\mathbf{x} \in \Omega$ ) are open intervals.

(i) In order that the chart  $(\chi, U)$  be  $v$ -adapted, it is necessary and sufficient that

$$\forall X \in U, \chi(l_X \cap U) \text{ has the form } I_{\mathbf{x}} \times \{\mathbf{x}\} \text{ for some } \mathbf{x} \in \Omega. \quad (19)$$

(ii) Moreover, if that is the case, then the  $v$ -adapted chart  $(\chi, U)$  is nice.

*Proof.* (i) If Condition (19) is satisfied, let  $l \in D_U$ . Thus,  $l \cap U \neq \emptyset$ , so let  $X \in l \cap U$ , hence  $l = l_X$ . Let  $\mathbf{x} \in \Omega$  be given by (19) for precisely the maximal integral curve  $l = l_X$ . For any  $Y \in l \cap U$ , we have thus  $\chi(Y) = (s, \mathbf{x})$  for some  $s \in I_{\mathbf{x}}$ . Hence, we have (13). Therefore, according to Definition 1,  $\chi$  is  $v$ -adapted. Conversely, assume that  $(\chi, U)$  is  $v$ -adapted. Let  $X \in U$  and set  $l \equiv l_X$ . Further, let  $\mathbf{x}$  be given by (13). That is, any point  $\mathbf{Y} \in \chi(l \cap U)$  has the form  $(s, \mathbf{x})$  for some  $s \in \mathbb{R}$ . Since moreover  $\chi(U)$  has the form (18), we have also  $\mathbf{Y} = (t, \mathbf{x}')$  for some  $\mathbf{x}' \in \Omega$  and some  $t \in I_{\mathbf{x}'}$ , so  $\mathbf{x} = \mathbf{x}'$  and  $s = t \in I_{\mathbf{x}}$ . Hence  $\chi(l \cap U) \subset I_{\mathbf{x}} \times \{\mathbf{x}\}$ . But also  $I_{\mathbf{x}} \times \{\mathbf{x}\} \subset \chi(l \cap U)$ . Indeed, setting  $l' \equiv \chi^{-1}(I_{\mathbf{x}} \times \{\mathbf{x}\})$ , we have  $l' \subset (l \cap U)$  because, as noted before the statement of this Proposition 1,  $l'$  is an integral curve of  $v$  that is included in  $U$  and that passes at  $X$ .

(ii) Assuming that the chart  $(\chi, U)$  obeys Condition (19) [and hence, by (i), is a  $v$ -adapted chart], let us show that the mapping  $\bar{\chi}$  defined by (14) is injective. Thus, let  $l, l' \in N_v$ , assume that both intersect  $U$ , and let  $\mathbf{x}$  and  $\mathbf{x}'$  be the images of  $l$  and  $l'$  by  $\bar{\chi}$ . This means, according to the definition (14), that we have (13), and similarly

$$\forall Y \in l' \cap U, \quad P_S(\chi(Y)) = \mathbf{x}'. \quad (20)$$

From Condition (19), we get thus:

$$\chi(l \cap U) = I_{\mathbf{x}} \times \{\mathbf{x}\} \quad \text{and} \quad \chi(l' \cap U) = I_{\mathbf{x}'} \times \{\mathbf{x}'\}. \quad (21)$$

Therefore, if  $\mathbf{x} = \mathbf{x}'$ , it is clear that  $l = l'$ . The proof is complete.  $\square$

The condition that the open set  $A \equiv \chi(U) \subset \mathbb{R}^{N+1}$  have the form (18) is fulfilled, in particular, if  $A$  is convex. So it is fulfilled if one restricts the chart  $\chi$  to  $\chi^{-1}(A)$  with  $A$  a convex open subset of  $\chi(U_0)$ . Unfortunately, what we have in a rather general situation is the following result, which does not ensure the applicability of Proposition 1:

**Theorem 0** [14]. *Suppose that some maximal integral curve  $l$  of the  $\mathcal{C}^\infty$  vector field  $v$  is closed in  $V$  and is not reduced to a point. Since  $v$  then does not vanish on  $l$ , let  $\chi : U \rightarrow I \times \Omega$  be a straightening-out chart in an open neighborhood  $U$  of some point of  $l$ . Then we have  $\chi(l \cap U) = I \times E$ , where  $E$  is a closed countable subset of  $\Omega$ , and any point  $X \in l \cap U$  is “transversally isolated”, i.e.  $\mathbf{x} \equiv P_S(\chi(X))$  is isolated in  $E$ .*

(Note that the set  $E$  is closed in  $\Omega$ , which is an open set in  $\mathbb{R}^N$ . So  $E$  is not necessarily closed in  $\mathbb{R}^N$ .) Assume that some straightening-out chart  $(\chi, U)$  is such that, for any  $X \in U$ , the maximal integral curve  $l_X$  is closed in  $V$ . From Point (i) in Proposition 1, we get that this is a  $v$ -adapted chart iff, in addition, for any point  $X \in U$ , the countable closed subset  $E_X$  of  $\Omega$ , whose existence is ensured by Theorem 0 for the curve  $l_X$ , is actually reduced to a point. We can further characterize this desired situation:

**Proposition 2.** *Let  $\chi : U_0 \rightarrow I \times \Omega_0$  be a straightening-out chart for the  $\mathcal{C}^\infty$  vector field  $v$ . Let  $U \subset U_0$  be an open set such that  $\chi(U)$  has the form (18). Assume that, for any  $X \in U$ , the maximal integral curve  $l_X$  is closed in  $V$ .*

- (i) *For any given  $X \in U$ , set  $\chi(X) = (s, \mathbf{x})$ . The connected component  $\lambda''$  of  $(s, \mathbf{x})$  in  $\lambda \equiv \chi(l_X \cap U)$  is equal to  $\lambda' \equiv I_{\mathbf{x}} \times \{\mathbf{x}\}$ .*
- (ii) *In order that  $(\chi, U)$  be a  $v$ -adapted chart, it is necessary and sufficient that, for any  $X \in U$ , the intersection  $l_X \cap U$  be a connected set.*
- (iii) *Let  $W$  be an open subset of  $U$  such that, for any  $X \in W$ ,  $l_X \cap U$  be a connected set. Then the restriction  $(\chi, W)$  is a nice  $v$ -adapted chart.*

*Proof.* (i) Since  $I_{\mathbf{x}}$  is an interval, the set  $\lambda' \equiv I_{\mathbf{x}} \times \{\mathbf{x}\}$  is connected. To show that  $\lambda'' = \lambda'$  is always true, we note first that,  $\chi(U)$  having the form (18),  $\chi(X) = (s, \mathbf{x})$  is such that  $s \in I_{\mathbf{x}}$ . Hence  $(s, \mathbf{x}) \in \lambda'$ , and since  $\lambda'$  is connected, we have  $\lambda' \subset \lambda''$ . To prove that in fact  $\lambda' = \lambda''$ , we will show that  $\lambda'$  is open and closed in  $\lambda''$ . We know from Theorem 0 that  $\lambda_0 \equiv \chi(l_X \cap U_0)$  has the form  $\lambda_0 = I \times E$ , where  $E$  is a subset of  $\Omega_0$  having only isolated points. Thus,  $\mathbf{x}$  being isolated in  $E$ , let  $r > 0$  be such that  $B \cap E = \{\mathbf{x}\}$ , where  $B \equiv B(\mathbf{x}, r)$  is the open ball of radius  $r$  in  $\mathbb{R}^N$ , centered at  $\mathbf{x}$ . Hence, we have

$$(I_{\mathbf{x}} \times B) \cap \lambda'' \subset (I_{\mathbf{x}} \times B) \cap \lambda_0 = (I_{\mathbf{x}} \times B) \cap (I \times E) = (I_{\mathbf{x}} \cap I) \times (B \cap E) = I_{\mathbf{x}} \times \{\mathbf{x}\} \equiv \lambda'. \quad (22)$$

We have also  $\lambda' \subset (I_{\mathbf{x}} \times B) \cap \lambda''$ , because  $\lambda' \subset I_{\mathbf{x}} \times B$  and  $\lambda' \subset \lambda''$ . So  $\lambda' = (I_{\mathbf{x}} \times B) \cap \lambda''$  is an open subset of  $\lambda''$ . On the other hand, if a sequence of points of  $\lambda'$  tends towards a limit in  $\lambda''$ , say  $(s_n, \mathbf{x}) \rightarrow (s', \mathbf{y}) \in \lambda''$ , then  $\mathbf{y} = \mathbf{x}$  and, since  $\lambda'' \subset \chi(U)$  which is given by (18), we have  $(s', \mathbf{x}) \in (\mathbb{R} \times \{\mathbf{x}\}) \cap \chi(U) = I_{\mathbf{x}} \times \{\mathbf{x}\} \equiv \lambda'$ , hence the limit  $(s', \mathbf{x})$  is in  $\lambda'$ , so that  $\lambda'$  is a closed subset of  $\lambda''$ . Being non-empty and an open-and-closed subset of the connected set  $\lambda''$ ,  $\lambda'$  is equal to  $\lambda''$ .

(ii) Since  $\chi : U_0 \rightarrow I \times \Omega_0$  is a bicontinuous mapping, it is of course equivalent to say that  $l_X \cap U$  or  $\lambda_X \equiv \chi(l_X \cap U)$  is connected. Therefore, (ii) follows immediately from (i) and from Statement (i) in Proposition 1.

(iii) Let  $X$  be any point in  $W$  and set  $\chi(X) = (s, \mathbf{x})$ . Since  $l_X \cap U$  is connected, we have  $\chi(l_X \cap U) = I_{\mathbf{x}} \times \{\mathbf{x}\}$  from (i). Thus,  $P_S(\chi(Y)) = \mathbf{x}$  is true for any  $Y \in l_X \cap U$ , hence a fortiori for any  $Y \in l_X \cap W$ . Hence the chart  $(\chi, W)$  is  $v$ -adapted. In a similar way, it is easy to adapt the proof of Statement (ii) in Proposition 1 to conclude that the  $v$ -adapted chart  $(\chi, W)$  is nice.  $\square$

**Proposition 3.** *Let  $\chi : U_0 \rightarrow I \times \Omega_0$  be a straightening-out chart for the  $\mathcal{C}^\infty$  vector field  $v$  and assume that, for some  $X \in U_0$ , the maximal integral curve  $l_X$  is closed. Let  $E$  be the closed countable subset of  $\Omega_0$  given by Theorem 0 for the maximal curve  $l \equiv l_X$ , thus  $\chi(l_X \cap U_0) = I \times E$ . Set  $\mathbf{x} \equiv P_S(\chi(X))$  and, as ensured by Theorem 0, let  $\Omega \subset \Omega_0$  be any open neighborhood of  $\mathbf{x}$  such that  $E \cap \Omega = \{\mathbf{x}\}$ . Let  $U \subset U_0$  be any open set such that  $\chi(U)$  has the form (18) with this set  $\Omega$ . Then  $l_X \cap U$  is connected,  $\chi(l_X \cap U) = I_{\mathbf{x}} \times \{\mathbf{x}\}$ .*

*Proof.* Since  $\chi : U_0 \rightarrow I \times \Omega_0$  is a bijection, we have

$$\begin{aligned} \chi(l_X \cap U) &= \chi((l_X \cap U_0) \cap U) = \chi(l_X \cap U_0) \cap \chi(U) \\ &= (I \times E) \cap \left( \bigcup_{\mathbf{x}' \in \Omega} I_{\mathbf{x}'} \times \{\mathbf{x}'\} \right) = \bigcup_{\mathbf{x}' \in \Omega} (I \cap I_{\mathbf{x}'} \times (E \cap \{\mathbf{x}'\})) \\ &= I_{\mathbf{x}} \times \{\mathbf{x}\}. \end{aligned} \tag{23} \quad \square$$

### 2.3 Intersections of straight lines with inverse images under the flow

Recall that the flow of the global vector field  $v$  on  $V$  is the mapping  $F : \mathcal{D} \rightarrow V$ ,  $(s, X) \mapsto F(s, X) \equiv C_X(s)$ , where  $\mathcal{D}$  is the domain of the flow  $F$ :

$$\mathcal{D} \equiv \bigcup_{X \in V} I_X \times \{X\} \subset \mathbb{R} \times V. \quad (24)$$

If  $v$  is  $\mathcal{C}^q$  ( $q \geq 1$ , possibly  $q = \infty$ ),  $\mathcal{D}$  is an open set in  $\mathbb{R} \times V$ , moreover  $F$  is  $\mathcal{C}^q$  on  $\mathcal{D}$  [12, 13].

**Proposition 4.** *Let  $U$  be an open subset of  $V$ . (i) For any  $X \in V$ , we have*

$$l_X \cap U = F_X(I'_{XU}) = F(I'_{XU} \times \{X\}), \quad (25)$$

where  $F_X \equiv F(\cdot, X) = C_X$  is defined on  $I_X \subset \mathbb{R}$ , and where

$$I'_{XU} \equiv F_X^{-1}(U) = \{s \in I_X; F(s, X) \in U\}. \quad (26)$$

(ii) Further, we have for any  $X \in V$ :

$$I'_{XU} \times \{X\} = (\mathbb{R} \times \{X\}) \cap \mathcal{D}_U = (I_X \times \{X\}) \cap \mathcal{D}_U, \quad (27)$$

where

$$\mathcal{D}_U \equiv F^{-1}(U) = \bigcup_{X \in V} I'_{XU} \times \{X\}. \quad (28)$$

(iii) Assume that  $v$  is  $\mathcal{C}^\infty$  and that, for some  $X \in V$ ,  $l_X$  is closed in  $V$  and not reduced to a point. Then (a)  $l_X$  is a submanifold of  $V$  and the mapping  $F_X : I_X \rightarrow l_X$  is a local diffeomorphism at any point  $s \in I_X$ . (b) Assume moreover that  $F_X$  is non-periodic. Then it is a (global) diffeomorphism of  $I_X$  onto  $l_X$ . The connected components of  $l_X \cap U$  are the images by  $F_X$  of the connected components of  $I'_{XU} \equiv F_X^{-1}(U) = F_X^{-1}(l_X \cap U)$ , which are open intervals of  $\mathbb{R}$ . In particular, in order that  $l_X \cap U$  be connected, it is necessary and sufficient that  $I'_{XU}$  be an open interval.

*Proof.* Points (i) and (ii) follow immediately from the definitions. Let us prove Point (iii). (a) Since the maximal integral curve  $l_X$  is closed in  $V$ , this is a submanifold of  $V$  [14]. Since  $l_X$  is not reduced to a point, the vector field  $v$  does not vanish on  $l_X$ . [If  $v(Y) = 0$  for some  $Y \in l_X$ , then we have

$I_Y = \mathbb{R}$  and  $C_Y(s) = Y \ \forall s \in \mathbb{R}$  from the uniqueness of the maximal integral curve, hence  $X = Y$  from the translation invariance (10).] Therefore,  $\frac{dF_X}{ds} = v(F_X(s)) \neq 0$  implies that  $F_X$  is a local diffeomorphism, at any point  $s \in I_X$ , between the one-dimensional manifolds  $I_X$  and  $l_X$ . (b) Since  $l_X$  is closed in  $V$ , not reduced to a point, and since  $F_X$  is non-periodic, it follows that  $F_X$  is injective [14]. In view of (a) and since  $F_X$  is surjective by the definition of  $l_X$ , it is a diffeomorphism of  $I_X$  onto  $l_X$ . Hence, the connected components  $l_j$  of  $l_X \cap U$  are the images by  $F_X$  of the connected components of the open set  $I'_{XU} \equiv F_X^{-1}(U) = F_X^{-1}(l_X \cap U) \subset \mathbb{R}$ , which are open intervals and make a finite or countable set  $\{I_j\}$  {Ref. [15], §(3.19.6)}.  $\square$

**An argument of transversality.** Assume that  $v$  does not vanish and that each maximal integral curve is non-periodic and is closed in  $V$ . Given an arbitrary point  $X \in V$ , let  $\chi : U_0 \rightarrow I \times \Omega_0$  be a straightening-out chart in the neighborhood of  $X$ , and let  $E$  be the closed countable subset of  $\Omega_0$ , having only isolated points, such that  $\chi(l_X \cap U_0) = I \times E$ . As Proposition 3 states: by restricting  $\chi$  to an open subset  $U \subset U_0$  for which  $\chi(U)$  has the form (18) with  $\Omega \subset \Omega_0$  any open neighborhood of  $\mathbf{x} \equiv P_S(\chi(X))$  such that  $E \cap \Omega = \{\mathbf{x}\}$ , we ensure that  $l_X \cap U$  is connected. From Point (iii) of Proposition 2, it follows that we will obtain a nice  $v$ -adapted chart by restricting  $\chi$  to an open neighborhood  $W \subset U$  of  $X$  — if it exists — such that, for any  $Y \in W$ ,  $l_Y \cap U$  be connected. To this purpose, we observe that, when the boundary of  $U$  is a regular hypersurface, then the same should be true for the boundary of the open set  $\mathcal{D}_U \equiv F^{-1}(U) \subset \mathbb{R} \times V$ : that boundary  $\Sigma_U \equiv \text{Fr}(\mathcal{D}_U)$  should “normally” be a hypersurface in  $\mathbb{R} \times V$ . I.e.,  $\Sigma_U$  should be a submanifold of codimension 1 of the  $(N + 2)$ -dimensional manifold  $\mathbb{R} \times V$ . Then, “generically”, a straight line  $\mathbb{R} \times \{X\} \subset \mathbb{R} \times V$  that intersects  $\Sigma_U$  is nowhere tangent to it, i.e. it is transverse to it at each intersection point. Thus, when  $Y$  is sufficiently close to  $X$ , the intersection points should be slightly displaced but should remain in the same number, hence the structure of  $I'_{YU} \times \{Y\} = (\mathbb{R} \times \{Y\}) \cap \mathcal{D}_U$  should be the same as for  $X$ . In particular, if  $I'_{XU}$  is an interval, i.e. (according to Proposition 4) if  $l_X \cap U$  is connected, then also  $I'_{YU}$  should be an interval, i.e., also  $l_Y \cap U$  should be connected.

To make this line of reasoning precise, we first introduce, for a general hypersurface  $\Sigma$  in  $\mathbb{R} \times V$ , the set  $S_\Sigma$  of the points  $X$  in  $V$  for which the straight

line  $\mathbb{R} \times \{X\}$  is tangent to  $\Sigma$  at one point at least. (Our aim is to use this when  $\Sigma$  is the boundary hypersurface  $\Sigma_U \equiv \text{Fr}(\mathcal{D}_U)$  introduced above.) Note that the tangent space to  $\mathbb{R} \times V$  at some point  $(s, X)$  is  $T_{(s,X)}(\mathbb{R} \times V) \simeq T_s\mathbb{R} \times T_XV \simeq \mathbb{R} \times T_XV$ . The tangent space to  $\mathbb{R} \times \{X\}$  at  $(s, X)$  is  $\mathbb{R}\xi$ , where  $\xi \equiv (1, 0_X)$ ,  $0_X$  being the zero element of  $T_XV$ . Thus we define

$$S_\Sigma \equiv \{X \in V; \exists s \in \mathbb{R} : (s, X) \in \Sigma \text{ and } (1, 0_X) \in T_{(s,X)}\Sigma\}. \quad (29)$$

However, we note that  $\mathcal{D}_U \equiv F^{-1}(U)$  is in general not a bounded domain of  $\mathbb{R} \times V$ , even if the open set  $U \subset V$  is bounded.<sup>3</sup> Therefore, even if  $X \in V$  is such that the straight line  $\mathbb{R} \times \{X\}$  is not tangent to the boundary hypersurface  $\Sigma_U$ , i.e.  $X \notin S_{\Sigma_U}$ , it may happen that  $\mathbb{R} \times \{X\}$  is “tangent to  $\Sigma_U$  at infinity”, in which case any line  $\mathbb{R} \times \{Y\}$ , however close  $Y$  can be to  $X$ , may have “new” intersections with the hypersurface  $\Sigma_U$ . For a general hypersurface  $\Sigma$  in  $\mathbb{R} \times V$ , in fact for *any subset*  $\Sigma$  of  $\mathbb{R} \times V$ , we thus introduce:

$$S_{\Sigma\infty} \equiv \{X \in V; \liminf_{r \rightarrow \infty} \inf_{|s| \geq r} d(X, \Sigma_s) = 0\}, \quad (30)$$

where, for any subset  $\mathcal{B}$  of  $\mathbb{R} \times V$ , we define  $\mathcal{B}_s$  to be its slice in  $V$  at  $s \in \mathbb{R}$ :

$$\mathcal{B}_s \equiv \{X \in V; (s, X) \in \mathcal{B}\} \subset V, \quad (31)$$

and where, given any distance  $d$  that generates the (metrizable) topology of  $V$ , one defines for any subset  $A$  of  $V$  and any point  $X \in V$ :  $d(X, A) \equiv \inf_{Z \in A} d(X, Z)$ .

## 2.4 Relevant theorems

**Theorem 1.** *Let  $\Sigma$  be a hypersurface of  $\mathbb{R} \times V$  that is closed in  $\mathbb{R} \times V$ . Let  $K = [\alpha, \beta]$  be a compact interval of  $\mathbb{R}$  ( $\alpha < \beta$ ). Suppose that, for some  $X_0 \in V$ , the intersection  $(K \times \{X_0\}) \cap \Sigma$  be a singleton  $(s_0, X_0)$  with  $\alpha < s_0 < \beta$  and that this intersection be transverse, i.e.  $X_0 \notin S_\Sigma$ . Then there is a neighborhood  $W$  of  $X_0$  such that, for any  $X \in W$ ,  $(K \times \{X\}) \cap \Sigma$  is a singleton  $\Phi(X)$  and this intersection is transverse. Moreover, the function*

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<sup>3</sup> For instance, consider a constant vector field  $v$  on  $V \equiv \mathbb{R}^n$ . Then we have simply  $F(s, X) = X + sv$ . Hence, for any  $X_0 \in V$ , its inverse image is the unbounded straight line  $F^{-1}(X_0) = \{(s, X) \in \mathbb{R} \times V; s \in \mathbb{R}, X = X_0 - sv\}$ . From this, one may deduce more general examples by applying a diffeomorphism.



$\Phi : W \rightarrow \Sigma$  is smooth.

*Proof.* Since  $(s_0, X_0) \in \Sigma$ , and since  $\Sigma$  is a submanifold of dimension  $N + 1$  of the  $(N + 2)$ -dimensional differentiable manifold  $\mathbb{R} \times V$ , there is a chart  $(\Psi, \mathcal{U})$  on  $\mathbb{R} \times V$ , with  $(s_0, X_0) \in \mathcal{U}$ , for which the function

$$g : \mathcal{U} \rightarrow \mathbb{R}, (s, X) \mapsto g(s, X) \equiv \Psi^{N+2}(s, X) \quad (32)$$

[the last component of  $\Psi(s, X)$ ] is such that

$$(s, X) \in \mathcal{U} \cap \Sigma \iff g(s, X) = 0 \quad (33)$$

{e.g. [11], §(16.8.3)}. Then, if  $(s, X) \in \mathcal{U} \cap \Sigma$ , a vector  $\eta = (a, u) \in T_{(s, X)}(\mathbb{R} \times V) \simeq \mathbb{R} \times T_X V$  is in the tangent space  $T_{(s, X)}\Sigma$ , iff  $dg_{(s, X)}(\eta) = 0$ . Since the intersection  $(K \times \{X_0\}) \cap \Sigma = \{(s_0, X_0)\}$  is transverse, we have

$$dg_{(s_0, X_0)}(\xi_0) \neq 0, \quad \xi_0 \equiv (1, 0_{T_{X_0}V}). \quad (34)$$

Since  $K$  is a neighborhood of  $s_0$ : replacing  $\mathcal{U}$  with a smaller neighborhood of  $(s_0, X_0)$  if necessary, we may assume that, if  $(s, X) \in \mathcal{U}$ , then  $s \in K$ . Consider a chart  $(\chi, U)$  of  $V$  in a neighborhood  $U$  of  $X_0$ , thus  $\chi(X) = \mathbf{X} = (x^\mu) \in \mathbb{R}^{N+1}$  for  $X \in U$ . The mapping  $\Xi : (s, X) \mapsto (s, \chi(X))$  is a local chart of  $\mathbb{R} \times V$  defined in the neighborhood  $\mathbb{R} \times U$  of  $(s_0, X_0)$ , while  $\Psi$  is also a local chart of  $\mathbb{R} \times V$ , defined in the neighborhood  $\mathcal{U}$  of  $(s_0, X_0)$ . Let

$$f(s, \mathbf{X}) \equiv g(\Xi^{-1}(s, \mathbf{X})) = g(s, \chi^{-1}(\mathbf{X})) \quad (35)$$

be the local expression of  $g$  in the chart  $\Xi$ . This function  $f$  is defined and  $\mathcal{C}^\infty$  on the open subset  $\mathcal{O} \equiv \Xi((\mathbb{R} \times U) \cap \mathcal{U})$  of  $\mathbb{R}^{N+2}$ . Note that  $\mathcal{O}$  contains  $(s_0, \mathbf{X}_0)$ , where  $\mathbf{X}_0 \equiv \chi(X_0)$ . With this local expression, we have for any vector  $\eta = (a, u) \in T_{(s, X)}(\mathbb{R} \times V)$ :

$$dg_{(s, X)}(\eta) = \frac{\partial f}{\partial s}(s, \mathbf{X}) a + \frac{\partial f}{\partial x^\mu}(s, \mathbf{X}) u^\mu, \quad (36)$$

where  $\mathbf{X} \equiv \chi(X)$ , and where  $(a, (u^\mu))$  ( $\mu = 0, \dots, N$ ) are the components of  $\eta$  in the product chart  $\Xi$ . From (33), we have

$$f(s_0, \mathbf{X}_0) = 0. \quad (37)$$

From (34) and (36) we have

$$dg_{(s_0, X_0)}(\xi_0) = \frac{\partial f}{\partial s}(s_0, \mathbf{X}_0) \neq 0. \quad (38)$$

We can thus apply the implicit function theorem: there is an open neighborhood  $A$  of  $\mathbf{X}_0$  and a unique smooth function  $\varphi : A \rightarrow \mathbb{R}$ , such that  $\varphi(\mathbf{X}_0) = s_0$  and that, for any  $\mathbf{X} \in A$ :

$$(\varphi(\mathbf{X}), \mathbf{X}) \in \mathcal{O}, \quad f(\varphi(\mathbf{X}), \mathbf{X}) = 0, \quad \frac{\partial f}{\partial s}(\varphi(\mathbf{X}), \mathbf{X}) \neq 0. \quad (39)$$

We define a smooth mapping  $\Phi$  by setting for any  $X \in W' \equiv \chi^{-1}(A) \subset V$ :

$$\Phi(X) \equiv (\varphi(\chi(X)), X). \quad (40)$$

Thus, for any  $X \in W'$ , with  $\mathbf{X} \equiv \chi(X) \in A$ , we have  $\Phi(X) = (\varphi(\mathbf{X}), \chi^{-1}(\mathbf{X})) = \Xi^{-1}(\varphi(\mathbf{X}), \mathbf{X}) \in \Xi^{-1}(\mathcal{O}) \subset \mathcal{U}$ . In particular,  $\Phi(X_0) = (s_0, X_0)$ . For any  $X \in W'$ , we have from (35), (39) and (40):  $g(\Phi(X)) = 0$ , thus  $\Phi(X) \in \Sigma \cap \mathcal{U}$ . As with (34) and (38), (39) means that the intersection  $\Phi(X) \in (K \times \{X\}) \cap \Sigma$  is transverse. The definition (40) entails also that  $\text{Pr}_2 \circ \Phi = \text{Id}_{W'}$ , where  $\text{Pr}_2 : \mathbb{R} \times V \rightarrow V$ ,  $(s, X) \mapsto X$ . Thus, the rank of  $\text{Pr}_2 \circ \Phi$  is  $\dim V = N + 1$ , and since the rank of  $\text{Pr}_2 \circ \Phi$  is not larger than that of  $\Phi$ , this latter is also  $N + 1 = \dim \Sigma$ , i.e.  $\Phi$  is a submersion. (All of this is true at any point  $X \in W'$ .) It follows that  $W'' \equiv \Phi(W')$  is an open set in the manifold  $\Sigma$  [11], §(16.7.5)}, hence is a neighborhood of  $(s_0, X_0) = \Phi(X_0)$  in  $\Sigma$ . We have

$$\forall (s, X) \in W'', \quad (s, X) = \Phi(X). \quad (41)$$

We claim that there is some neighborhood  $W \subset W'$  of  $X_0$ , such that

$$\forall X \in W, \quad [s \in K \text{ and } (s, X) \in \Sigma] \implies [(s, X) = \Phi(X)]. \quad (42)$$

Note that, if  $W \subset W'$ , the reverse implication is also true for any  $X \in W$ , by the construction of the mapping  $\Phi$ . Thus, if (42) is true,  $W$  is as stated by Theorem 1. We will reason *ab absurdo*. If it does not exist such a neighborhood  $W$ , then, for any integer  $n > 0$ , one can find  $s_n \in K$  and  $X_n \in W' \cap B(X_0, 1/n)$  such that  $(s_n, X_n) \in \Sigma$  and  $(s_n, X_n) \neq \Phi(X_n)$ . Thus  $X_n \rightarrow X_0$  and, by extraction in the compact set  $K$ , we may assume that  $s_n$  has a limit  $s \in K$ , so that  $(s_n, X_n) \rightarrow (s, X_0)$ . If it happened that  $s = s_0$ , then, since  $W''$  is a neighborhood of  $(s_0, X_0)$  and  $(s_n, X_n) \rightarrow (s, X_0)$ , we would have  $(s_n, X_n) \in W''$  for large enough  $n$ , hence  $(s_n, X_n) = \Phi(X_n)$  from (41), which is a contradiction. Thus  $s \neq s_0$ . But since  $\Sigma$  is closed, we have  $(s, X_0) \in \Sigma$ , which contradicts the assumption that  $(K \times \{X_0\}) \cap \Sigma = \{(s_0, X_0)\}$ . This completes the proof of Theorem 1.  $\square$

**Theorem 2.** *Let  $\mathcal{U}$  be an open domain in  $\mathbb{R} \times V$ , whose boundary  $\Sigma$  be a hypersurface of  $\mathbb{R} \times V$ . Assume that, for some  $X \in V$ , all intersections of the straight line  $\mathcal{L} \equiv \mathbb{R} \times \{X\}$  with  $\Sigma$  are transverse. Then the boundary of  $\mathcal{L} \cap \mathcal{U}$  in  $\mathcal{L}$  is  $\text{Fr}_{(\mathcal{L})}(\mathcal{L} \cap \mathcal{U}) = \mathcal{L} \cap \Sigma$ .*

*Proof.* The boundary of a subset is of course relative to which containing set is considered. Here, the boundary  $\Sigma$  of the open set  $\mathcal{U} \subset \mathbb{R} \times V$  is relative to the whole manifold  $\mathcal{V} \equiv \mathbb{R} \times V$ , i.e.  $\Sigma \equiv \overline{\mathcal{U}} \cap \overline{\mathcal{C}\mathcal{U}}$ , where the upper bar  $\bar{\cdot}$  denotes the adherence in  $\mathcal{V}$  and  $\mathcal{C}$  the complementary set in  $\mathcal{V}$ . Thus

$$\Sigma = \overline{\mathcal{U}} \setminus \mathcal{U}, \quad (43)$$

since  $\mathcal{U}$  is open. The boundary of some subset  $\mathcal{B} \subset \mathcal{L}$  in  $\mathcal{L}$  (or relative to  $\mathcal{L}$ ) is  $\text{Fr}_{(\mathcal{L})}(\mathcal{B}) \equiv \overline{\mathcal{B}}^{\mathcal{L}} \cap \overline{\mathcal{C}_{\mathcal{L}}\mathcal{B}}^{\mathcal{L}}$ , where  $\overline{\mathcal{B}}^{\mathcal{L}}$  denotes the adherence of  $\mathcal{B}$  in  $\mathcal{L}$ , and where  $\mathcal{C}_{\mathcal{L}}\mathcal{B} \equiv \mathcal{L} \setminus \mathcal{B}$  is the complementary set of  $\mathcal{B}$  in  $\mathcal{L}$ . However, here  $\mathcal{L} \equiv \mathbb{R} \times \{X\}$  is closed in  $\mathcal{V} = \mathbb{R} \times V$ , hence we have  $\overline{\mathcal{B}}^{\mathcal{L}} = \overline{\mathcal{B}}$ , the adherence in the whole set  $\mathcal{V}$ . Thus

$$\text{Fr}_{(\mathcal{L})}(\mathcal{L} \cap \mathcal{U}) \equiv \overline{\mathcal{L} \cap \mathcal{U}}^{\mathcal{L}} \cap \overline{\mathcal{C}_{\mathcal{L}}(\mathcal{L} \cap \mathcal{U})}^{\mathcal{L}} = \overline{\mathcal{L} \cap \mathcal{U}} \cap \overline{\mathcal{C}_{\mathcal{L}}\mathcal{U}}. \quad (44)$$

Again because  $\mathcal{L}$  is closed in  $\mathcal{V}$ , we have  $\overline{\mathcal{L} \cap \mathcal{U}} \subset \mathcal{L} \cap \overline{\mathcal{U}}$ . We will show that we have exactly

$$\overline{\mathcal{L} \cap \mathcal{U}} = \mathcal{L} \cap \overline{\mathcal{U}}. \quad (45)$$

We shall in fact show that

$$\mathcal{L} \cap \Sigma \subset \overline{\mathcal{L} \cap \mathcal{U}}. \quad (46)$$

Since (43) implies that  $(\mathcal{L} \cap \overline{\mathcal{U}}) \setminus \overline{\mathcal{L} \cap \mathcal{U}} \subset (\mathcal{L} \cap \overline{\mathcal{U}}) \setminus (\mathcal{L} \cap \mathcal{U}) = \mathcal{L} \cap \Sigma$ , this will prove that  $\mathcal{L} \cap \overline{\mathcal{U}} \subset \overline{\mathcal{L} \cap \mathcal{U}}$ , whence (45).

To prove (46), consider an arbitrary point  $p_0 = (s_0, X) \in \mathcal{L} \cap \Sigma$ . As in the proof of Theorem 1, let  $(\Psi, \mathcal{W})$  be a chart on  $\mathcal{V}$ , with  $p_0 \in \mathcal{W}$ , such that  $p \in \mathcal{W} \cap \Sigma \Leftrightarrow g(p) = 0$ , where  $g = \Psi^n$ , with  $n = \dim(\mathcal{V}) = N + 2$ . Since we assume that the intersection  $p_0 \in \mathcal{L} \cap \Sigma$  is transverse, we have again (34). Hence, there is an interval  $J = ]s_0 - r, s_0 + r[$  in which  $s = s_0$  is the only zero of the smooth function  $\varphi(s) \equiv g(s, X)$ . (That is,  $p_0$  is the only intersection of  $J \times \{X\}$  with  $\Sigma$ .) Thus, we may assume that, say,  $g(s, X) > 0$  for  $s_0 < s < s_0 + r$ , so that

$$g(s, X) < 0 \text{ for } s_0 - r < s < s_0. \quad (47)$$

Replacing  $\Psi(p)$  by  $\Psi(p) - \Psi(p_0)$ , we may assume that  $\Psi(p_0) = \mathbf{0}_{\mathbb{R}^n}$ . There is some open ball  $W = B(\mathbf{0}, r) \subset \Psi(W)$ . Replacing  $W$  by  $\Psi^{-1}(W)$ , we have that  $\mathcal{W}_+ \equiv \{p \in \mathcal{W}; x^n \equiv \Psi^n(p) > 0\}$  is just  $\mathcal{W}_+ = \Psi^{-1}(W_+)$ , where  $W_+ \equiv \{\mathbf{P} \in W; x^n > 0\}$ , hence  $\mathcal{W}_+$  is non-empty and connected as is  $W_+$ . The same is true for  $\mathcal{W}_- \equiv \{p \in \mathcal{W}; \Psi^n(p) < 0\}$ . Since  $p_0 \in \Sigma \subset \overline{\mathcal{U}}$ , and since  $\mathcal{W}$  is a neighborhood of  $p_0$ , we have  $\mathcal{U} \cap \mathcal{W} \neq \emptyset$ , so let  $p'_0 \in \mathcal{U} \cap \mathcal{W}$ . Because  $\mathcal{W}$  is the disjointed union  $\mathcal{W} = \mathcal{W}_+ \cup \mathcal{W}_- \cup (\mathcal{W} \cap \Sigma)$ , and because  $p'_0 \in \mathcal{U}$  cannot belong to  $\Sigma = \overline{\mathcal{U}} \setminus \mathcal{U}$ , we have either  $p'_0 \in \mathcal{W}_+$  or  $p'_0 \in \mathcal{W}_-$ . Let us assume that, for instance,  $p'_0 \in \mathcal{W}_-$ , so that  $\mathcal{U} \cap \mathcal{W}_- \neq \emptyset$ . It follows that  $\mathcal{W}_- \subset \mathcal{U} \cap \mathcal{W}$ , for otherwise the connected set  $\mathcal{W}_-$  would intersect both  $\mathcal{U}$  and  $\overline{\mathcal{U}}$ , hence would intersect the boundary  $\Sigma$  — which is impossible, since we have  $x^n = 0$  in  $\Sigma$ , not  $x^n < 0$ . Therefore, we get from (47) that  $]s_0 - r, s_0[ \times \{X\} \subset \mathcal{U} \cap \mathcal{W}$ , so that  $p_0 \equiv (s_0, X) \in \overline{\mathcal{L} \cap \mathcal{U}}$ . This proves (46).

Combining (44) and (45) gives us:

$$\text{Fr}_{(\mathcal{L})}(\mathcal{L} \cap \mathcal{U}) = \mathcal{L} \cap \overline{\mathcal{U}} \cap \overline{\mathcal{C}_{\mathcal{L}} \mathcal{U}}. \quad (48)$$

But  $\mathcal{C}_{\mathcal{L}} \mathcal{U} = \mathcal{L} \cap \mathcal{C} \mathcal{U}$  is closed in  $\mathcal{V}$ , for  $\mathcal{L}$  is closed and  $\mathcal{U}$  is open. Hence  $\mathcal{L} \cap \overline{\mathcal{C}_{\mathcal{L}} \mathcal{U}} = \mathcal{L} \cap \mathcal{C}_{\mathcal{L}} \mathcal{U} = \mathcal{L} \cap \mathcal{C} \mathcal{U} = \mathcal{L} \cap \overline{\mathcal{C} \mathcal{U}}$ . Therefore, (48) rewrites as

$$\text{Fr}_{(\mathcal{L})}(\mathcal{L} \cap \mathcal{U}) = \mathcal{L} \cap \overline{\mathcal{U}} \cap \overline{\mathcal{C} \mathcal{U}} \equiv \mathcal{L} \cap \Sigma, \quad (49)$$

which proves Theorem 2.  $\square$

**Remark 2.1.** With small and straightforward modifications, the foregoing proof shows the result in the much more general case that  $\mathbb{R} \times V$  (with  $V$  a differentiable manifold) is replaced by a general differentiable manifold  $\mathcal{V}$  and  $\mathcal{L}$  is the range  $\mathcal{L} = C(J)$ , assumed closed, of a smooth curve  $C : J \rightarrow \mathcal{V}$  (with  $J$  an interval of  $\mathbb{R}$ ), such that all intersections  $p \in \mathcal{L} \cap \mathcal{V}$  are transverse. It is easy to see that the latter assumption is necessary.

**Remark 2.2.** In the course of the proof, the following intuitively obvious result was proved: Suppose that the line  $\mathcal{L} = C(J)$  intersects transversely at point  $p = C(s_0)$  the boundary  $\Sigma$ , assumed to be a regular hypersurface, of some open domain  $\mathcal{U} \subset \mathcal{V}$ . Then, among the two parts of  $\mathcal{L}$ :  $s < s_0$  and  $s > s_0$ , at least one is such that, when  $s$  is close enough to  $s_0$ , we have  $C(s) \in \mathcal{U}$ .

**Theorem 3.** *Let  $U$  be an open subset of  $V$ . (i) Assume that  $\overline{F^{-1}(U)} \subset \mathcal{D}$ , with  $\mathcal{D}$  the domain of the flow  $F$ . (This is true, in particular, if the flow is complete, i.e.  $\mathcal{D} = \mathcal{V}$ .) Then we have:*

$$\text{Fr}(F^{-1}(U)) = F^{-1}(\text{Fr}(U)). \quad (50)$$

(ii) *If  $\text{Fr}(U)$  is a hypersurface of  $V$ , then  $F^{-1}(\text{Fr}(U))$  is a hypersurface of  $\mathcal{V} \equiv \mathbb{R} \times V$ .*

*Proof of Point (i).* First, consider the more general context that  $\mathcal{V}$  and  $V$  are merely metric spaces and  $F : \mathcal{D} \rightarrow V$  is merely a continuous map, with  $\mathcal{D}$  an open subset of  $\mathcal{V}$ . Then, if  $\overline{F^{-1}(U)} \subset \mathcal{D}$ , we have

$$\text{Fr}(F^{-1}(U)) \subset F^{-1}(\text{Fr}(U)). \quad (51)$$

Indeed, since  $U$  is open in  $V$ , we have  $\text{Fr}(U) = \overline{U}^V \setminus U$ , as with Eq. (43). Since  $F$  is continuous on  $\mathcal{D}$ ,  $F^{-1}(U)$  is open in  $\mathcal{D}$ ; hence,  $\mathcal{D}$  being an open subset of  $\mathcal{V}$ ,  $F^{-1}(U)$  is open in  $\mathcal{V}$ . Therefore, we have similarly  $\text{Fr}(F^{-1}(U)) = \overline{F^{-1}(U)} \setminus F^{-1}(U)$ . Again because  $F$  is continuous on  $\mathcal{D}$ , we have  $\overline{F^{-1}(U)}^{\mathcal{D}} \subset F^{-1}(\overline{U}^V)$ . But  $\overline{F^{-1}(U)}^{\mathcal{D}} = \overline{F^{-1}(U)}$  since  $\overline{F^{-1}(U)} \subset \mathcal{D}$ . Thus  $\text{Fr}(F^{-1}(U)) \subset F^{-1}(\overline{U}^V) \setminus F^{-1}(U)$ , whence (51).

To prove the reverse inclusion, we consider any  $p = (s, X) \in F^{-1}(\text{Fr}(U))$ , thus  $Y \equiv F(s, X) \in \text{Fr}(U)$ , and we will show that  $p \in \text{Fr}(F^{-1}(U))$ . There exists an open neighborhood  $\mathcal{U}$  of  $p$ , having the form  $\mathcal{U} = J \times W$ , with  $J$  an open interval containing  $s$  and 0, and with  $W$  an open neighborhood of  $X$  in  $V$ , such that  $\mathcal{U} \subset \mathcal{D}$  {Ref. [12], §(18.2.5)}. For all  $t \in J$ ,  $F_t \equiv F(t, \cdot)$  is a homeomorphism of  $W$  onto  $W_t \equiv F_t(W)$ , moreover  $F_{-t}$  is defined over  $W_t$  and is the inverse homeomorphism of  $F_t$  [these two points result from (10)] {Ref. [12], §(18.2.8)}. (We note for the proof of Point (ii) that  $F_t$ , as well as  $F_{-t}$ , is of class  $\mathcal{C}^q$  if the vector field  $v$  is itself  $\mathcal{C}^q$  [12].) Now we consider any neighborhood  $\mathcal{U}'$  of  $p$  and we show that it intersects both  $F^{-1}(U)$  and  $\mathbb{C}F^{-1}(U)$ . We may assume that  $\mathcal{U}' \subset \mathcal{U}$  and has the form  $\mathcal{U}' = J' \times W'$ , with  $J' \subset J$  an open interval containing  $s$ , and with  $W' \subset W$  an open neighborhood of  $X$  in  $V$ . Thus  $F_s(W')$  is an open neighborhood of  $Y = F_s(X)$ . Since  $Y \in \text{Fr}(U)$ , both  $F_s(W') \cap U$  and  $F_s(W') \cap \mathbb{C}U$  are non-empty, so let  $Y' \in F_s(W') \cap U$  and  $Y'' \in F_s(W') \cap \mathbb{C}U$ . Therefore,  $X' \equiv F_{-s}(Y') \in W'$ . Thus,  $F(s, X') = Y' \in U$ , and also  $(s, X') \in J' \times W'$ , so that  $(s, X') \in (J' \times W') \cap F^{-1}(U)$ . In just the

same way, with  $X'' \equiv F_{-s}(Y'')$ , we get that  $(s, X'') \in (J' \times W') \cap F^{-1}(\mathbb{C}U)$ . Since  $F^{-1}(\mathbb{C}U) = \mathbb{C}_{\mathcal{D}}F^{-1}(U) \subset \mathbb{C}F^{-1}(U) \equiv \mathcal{V} \setminus F^{-1}(U)$ , we have shown that any neighborhood  $\mathcal{U}'$  of  $p$  intersects both  $F^{-1}(U)$  and  $\mathbb{C}F^{-1}(U)$ , thus  $p \in \text{Fr}(F^{-1}(U))$ . Therefore, we have indeed  $F^{-1}(\text{Fr}(U)) \subset \text{Fr}(F^{-1}(U))$ . Together with (51), this proves (50).

*Proof of Point (ii).* Let  $p = (s, X) \in F^{-1}(\text{Fr}(U)) \subset \mathcal{V}$ . Define  $J$  (with  $s \in J$ ),  $W \subset V$  (with  $X \in W$ ),  $\mathcal{U} = J \times W$ , and  $F_t$  ( $t \in J$ ) just as at the beginning of the foregoing paragraph. Since  $S \equiv \text{Fr}(U)$  is assumed to be a hypersurface of  $V$ , and since by hypothesis  $Y \equiv F(s, X) \in S$ , let  $\chi : Y' \mapsto \mathbf{Y} \equiv (y^1, \dots, y^{n-1})$  be a chart of  $V$  (with  $n - 1 = \dim(\mathcal{V}) - 1 = \dim(V)$ ), defined in the neighborhood of  $Y$ , and such that

$$Y' \in S \cap \text{Dom}(\chi) \Leftrightarrow y^{n-1} \equiv \chi^{n-1}(Y') = 0. \quad (52)$$

We may assume that  $\text{Dom}(\chi)$ , the domain of  $\chi$ , is just  $W_s \equiv F_s(W)$ . Then the mapping

$$\Xi : \mathcal{U} = J \times W \rightarrow \mathbb{R}^n, \quad (t, X') \mapsto (t, \chi(F_s(X'))) = (t, \mathbf{Y}) \quad (53)$$

is a chart of  $\mathcal{V}$  in the neighborhood of  $p$ . Consider the mapping

$$\Psi : \mathcal{U}' \equiv F^{-1}(W_s) \rightarrow \mathbb{R}^n, \quad (t, X') \mapsto (t, \chi(F(t, X'))). \quad (54)$$

(Note that  $\mathcal{U}'$  is an open neighborhood of  $p$ .) We have (cf. Eq. (10)):

$$F(t, X') = F(s + u, X') = F(u, F(s, X')) = F(u, F_s(X')), \quad u \equiv t - s. \quad (55)$$

Hence, the local expression of  $\Psi$  in the chart  $\Xi$  is:

$$\mathbf{G}(t, \mathbf{Y}) \equiv \Psi(\Xi^{-1}(t, \mathbf{Y})) = (t, \mathbf{Z}(t, \mathbf{Y})) \quad (56)$$

with

$$\mathbf{Z}(t, \mathbf{Y}) = (z^1, \dots, z^{n-1}) \equiv \chi(F(u, \chi^{-1}(\mathbf{Y}))). \quad (57)$$

Let  $\mathbf{v} = (v^1, \dots, v^{n-1}) = \mathbf{v}(\mathbf{Y})$  be the local expression of  $v$  in the chart  $\chi$ . Thus  $\mathbf{Z}$  is the value at  $u \equiv t - s$  of the solution of  $\frac{d\mathbf{Z}'}{dw} = \mathbf{v}(\mathbf{Z}'(w))$ ,  $\mathbf{Z}'(w = 0) = \mathbf{Y}$ . Hence we have, uniformly w.r.t.  $(t, \mathbf{Y})$  in some neighborhood of  $(s, \mathbf{Y}_0) \equiv \Xi(p)$ :

$$\mathbf{Z}(t, \mathbf{Y}) = \mathbf{Y} + (t - s)\mathbf{v}(\mathbf{Y}) + O(u^2). \quad (58)$$

The Jacobian matrix of  $\mathbf{G}(t, \mathbf{Y}) = (t, \mathbf{Z}(t, \mathbf{Y}))$  at point  $(s, \mathbf{Y}_0)$  is therefore:

$$J = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ v^1(\mathbf{Y}_0) & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ v^{n-1}(\mathbf{Y}_0) & 0 & 0 & \dots & 1 \end{pmatrix}, \quad (59)$$

a triangular matrix with 1 on the diagonal, so  $\det J = 1$ . It follows that  $\Psi$  also is a chart of  $\mathcal{V}$  in some neighborhood  $\mathcal{U}'' \subset \mathcal{U} \cap \mathcal{U}'$  of  $p$ . When  $(t, X') \in \mathcal{U}''$ , we have from (52) and (54):

$$(t, X') \in F^{-1}(S) \Leftrightarrow z^{n-1} \equiv \Psi^n(t, X') \equiv \chi^{n-1}(F(t, X')) = 0. \quad (60)$$

Hence,  $F^{-1}(S)$  is a hypersurface of  $\mathcal{V}$ . The proof of Theorem 3 is complete.  $\square$

**Remark 3.1.** From (54) and (56), we have also  $\mathbf{Z}(t, \mathbf{Y}) = \chi(F(\Xi^{-1}(t, \mathbf{Y})))$ , thus  $\mathbf{Z}(t, \mathbf{Y})$  is the local expression of  $F$  in the charts  $\Xi$  on  $\mathcal{V}$  and  $\chi$  on  $V$ . Equation (59) shows also that the Jacobian matrix of  $\mathbf{Z}$  at  $(s, \mathbf{Y}_0) \equiv \Xi(p)$  has rank  $n - 1 = \dim V$ . [Relation (52), hence the fact that  $p \in F^{-1}(S)$ , are not used to get this; hence this is true at any point  $p \in \mathcal{D} = \text{Dom}(F)$ .] Thus,  $F$  is a submersion. Hence, it is transversal to any submanifold of  $V$ . Point (ii) of Theorem 3 follows also from this [16].

**Proposition 5.** *Let  $\Sigma$  be a subset of  $\mathbb{R} \times V$ . If  $X \in V \setminus S_{\Sigma\infty}$  [cf. (30)], there is a neighborhood  $W$  of  $X$  and a real  $R > 0$  such that*

$$(Y \in W \text{ and } |s| \geq R) \Rightarrow (s, Y) \notin \Sigma. \quad (61)$$

*Proof.* Clearly,  $f(X) \equiv \lim_{r \rightarrow \infty} \inf_{|s| \geq r} d(X, (\Sigma)_s)$  is well defined for any  $X \in V$  and verifies  $0 \leq f(X) \leq +\infty$ . Hence, if  $X \notin S_{\Sigma\infty}$  it means that  $f(X) > 0$ , or equivalently that there exists  $R > 0$  and  $\delta > 0$  such that

$$|s| \geq R \Rightarrow d(X, \Sigma_s) \geq \delta, \quad (62)$$

hence  $d(X, Z) \geq \delta$  is true for any  $Z \in \Sigma_s$  if  $|s| \geq R$ . We deduce from this that, if  $Y \in V$  verifies  $d(X, Y) < \delta/2$ , and if  $|s| \geq R$ , we have  $d(Y, Z) > \delta/2$  for any  $Z \in \Sigma_s$ , hence  $d(Y, \Sigma_s) \geq \delta/2$ . (Thus,  $V \setminus S_{\Sigma\infty}$  is open in  $V$ , in other words  $S_{\Sigma\infty}$  is closed.) In particular, if  $d(X, Y) < \delta/2$  and  $|s| \geq R$ , then  $Y \notin \Sigma_s$ , i.e.  $(s, Y) \notin \Sigma$ .  $\square$

**Theorem 4.** *Let  $U$  be an open subset of  $V$  and set  $\mathcal{D}_U \equiv F^{-1}(U)$  and  $\Sigma_U \equiv \text{Fr}(\mathcal{D}_U)$ . Assume that  $\Sigma_U$  is a hypersurface of  $\mathcal{V} \equiv \mathbb{R} \times V$  which is closed in  $\mathcal{V}$ , that for some  $X \in U$ , the open set  $I'_{XU}$  is a bounded interval, and that  $X \notin (S_{\Sigma_U} \cup S_{\Sigma_U \infty})$ .*

*(i) There is a neighborhood  $W$  of  $X$ ,  $W \subset U$ , such that for  $Y \in W$ , also  $I'_{YU}$  is a bounded interval. (ii) If  $v$  does not vanish and all maximal integral curves are closed in  $V$  and non-periodic, then  $l_Y \cap U$  is connected for any  $Y \in W$ . (iii) If moreover  $\chi(U)$  has the form (18) and there is a straightening-out chart  $(\chi, U_0)$  with  $U_0 \supset U$ , then the restriction of  $\chi$  to  $W$  is a nice  $v$ -adapted chart.*

*Proof.* (i) Since  $X \notin S_{\Sigma_U \infty}$ , by Proposition 5 there is a neighborhood  $W_0$  of  $X$  and a real  $R > 0$  such that

$$(Y \in W_0 \text{ and } |s| \geq R) \Rightarrow (s, Y) \notin \Sigma_U. \quad (63)$$

Since  $I'_{XU}$  is assumed to be a bounded interval (and  $I'_{XU} \neq \emptyset$  since  $0 \in I'_{XU}$ ), let  $I'_{XU} = ]a, b[$ , with  $a, b \in \mathbb{R}$ ,  $a < b$ . In (63) we may assume that  $-R < a < b < R$ . By Proposition 4, we have

$$I'_{XU} \times \{X\} = ]a, b[ \times \{X\} = (\mathbb{R} \times \{X\}) \cap \mathcal{D}_U. \quad (64)$$

Since by assumption  $X \notin S_{\Sigma_U}$ , we get by Theorem 2 (setting  $\mathcal{L} \equiv \mathbb{R} \times \{X\}$ ):

$$\{(a, X), (b, X)\} = \text{Fr}_{(\mathcal{L})}(\mathcal{L} \cap \mathcal{D}_U) = \mathcal{L} \cap \Sigma_U. \quad (65)$$

Thus, considering the compact intervals  $K_1 \equiv [-R, \frac{a+b}{2}]$  and  $K_2 \equiv [\frac{a+b}{2}, R]$ , we have

$$(K_1 \times \{X\}) \cap \Sigma_U = \{(a, X)\}, \quad (K_2 \times \{X\}) \cap \Sigma_U = \{(b, X)\}. \quad (66)$$

Therefore, by Theorem 1, there are neighborhoods  $W_1$  and  $W_2$  of  $X$ , and smooth functions  $\Phi_j : W_j \rightarrow \Sigma_U$  ( $j = 1, 2$ ), such that:

$$Y \in W_j \Rightarrow (K_j \times \{Y\}) \cap \Sigma_U = \{\Phi_j(Y)\} \quad (j = 1, 2), \quad (67)$$

with transverse intersection. We have  $\Phi_j(Y) = (\varphi_j(Y), Y)$ , with  $\varphi_j : W_j \rightarrow \mathbb{R}$  a smooth function. From (66) and (67), we get:  $\varphi_1(X) = a$  and  $\varphi_2(X) = b$ . Thus  $\varphi_1(X) = a \neq b = \varphi_2(X)$ ; hence, by considering a small enough neighborhood  $W$  of  $X$ , with  $W \subset W_0 \cap W_1 \cap W_2$ , we get  $\varphi_1(Y) \neq \varphi_2(Y)$  for  $Y \in W$ . With (63) and (67), this implies that

$$Y \in W \Rightarrow (\mathbb{R} \times \{Y\}) \cap \Sigma_U = \{\Phi_1(Y), \Phi_2(Y)\}. \quad (68)$$



Another application of Theorem 2 proves that

$$\text{Fr}_{(\mathbb{R} \times \{Y\})} ((\mathbb{R} \times \{Y\}) \cap \mathcal{D}_U) = (\mathbb{R} \times \{Y\}) \cap \Sigma_U. \quad (69)$$

Together with (68), this implies that  $I'_{YU}$ , the open subset of  $\mathbb{R}$  such that  $I'_{YU} \times \{Y\} = (\mathbb{R} \times \{Y\}) \cap \mathcal{D}_U$ , has boundary  $\{\varphi_1(Y), \varphi_2(Y)\}$ . Therefore,  $I'_{YU} = ]\varphi_1(Y), \varphi_2(Y)[$ .

(ii) This follows from Point (iii) in Proposition 4.

(iii) This follows from Point (iii) in Proposition 2.  $\square$

## 2.5 Adapted charts and “normal” vector fields

With Theorem 4, we formalized our transversality argument in Subsect. 2.3 to investigate the problem of the existence, in the neighborhood of any point  $X \in V$ , of a nice  $v$ -adapted chart. Assuming that  $v$  does not vanish and that all maximal integral curves are closed in  $V$  and non-periodic, let us check if Theorem 4 applies. Due to Proposition 4,  $I'_{XU} = F_X^{-1}(l_X \cap U)$  is an interval iff  $l_X \cap U$  is connected, and  $I'_{XU}$  is bounded if  $U$  is relatively compact. As shown by Proposition 3, the assumption that  $l_X \cap U$  is connected may be fulfilled by starting from a straightening-out chart  $\chi : U_0 \rightarrow I \times \Omega_0$  in the neighborhood of the arbitrary point  $X \in V$  and by restricting  $\chi$  to an open subset  $U \subset U_0$  such that  $\chi(U)$  has the form (18) with  $\Omega \subset \Omega_0$  a small enough open neighborhood of  $\mathbf{x} \equiv P_S(\chi(X))$ .

As shown by Theorem 3, the assumption that  $\Sigma_U$  is a hypersurface of  $\mathcal{V}$  that is closed in  $\mathcal{V}$  is fulfilled, in particular, if the boundary of the open set  $U \subset V$  is itself a hypersurface of  $V$  that is closed in  $V$ , and if moreover  $\overline{F^{-1}(U)} \subset \mathcal{D}$ . The latter inclusion is true, in particular, if  $\mathcal{D} = \mathcal{V}$ , i.e., if the vector field  $v$  is complete (in other words, if every maximal integral curve of  $v$  is defined over the whole real line). Actually, this does not restrict in any way the set of the maximum integral curves,  $N_v \equiv \{l_X; X \in V\}$  (the “congruence of world lines”, in the physical context with  $N = 3$ ). Indeed, there always exists a smooth function  $\lambda : V \rightarrow \mathbb{R}_+$ , such that the vector field  $\lambda v$  is complete, moreover the mappings  $C_X$  corresponding to the maximal integral curves of  $\lambda v$  are mere reparameterizations of those of  $v$ , so that the curves  $l_X$  themselves are unchanged [17]. Thus, the assumption that  $\Sigma_U$  is a hypersurface of  $\mathcal{V}$  that is closed in  $\mathcal{V}$  is not a restrictive one.

The assumption “ $X \notin (S_{\Sigma_U} \cup S_{\Sigma_U \infty})$ ” means that the straight line  $\mathbb{R} \times \{X\}$  is not tangent to the hypersurface  $\Sigma_U$ , and is not “tangent to it at infinity”. For a given hypersurface  $\Sigma_U$ , the points thus excluded form a kind of apparent contour (of that hypersurface  $\Sigma_U$ ), having “normally” measure zero in  $V$ , hence this is true for a “generic” point  $X$ . However, here the hypersurface  $\Sigma_U = F^{-1}(\text{Fr}(U))$  of  $\mathbb{R} \times V$  depends on the selected neighborhood  $U$  of the given point  $X \in V$ . There is much freedom in the choice of this neighborhood, since it is merely required to have a regular boundary and have the form (18) with  $\Omega$  a small enough open neighborhood of  $\mathbf{x} \equiv P_S(\chi(X))$ . If it turns out that  $X \in (S_{\Sigma_U} \cup S_{\Sigma_U \infty})$  for some  $U$  satisfying these requirements, then a slightly deformed neighborhood  $U'$  does also satisfy them, but the boundary  $\Sigma_{U'}$  is also deformed w.r.t.  $\Sigma_U$ . Hence, it seems plausible that, due to this freedom, there always exists  $U$  satisfying these requirements and such that  $X \notin (S_{\Sigma_U} \cup S_{\Sigma_U \infty})$  — unless  $v$  has some “pathology” that we were not able to describe in a more explicit way.

Thus, for any point  $X \in V$ , the assumptions of Theorem 4 should be fulfilled in a suitable neighborhood  $U$  of  $X$  if the vector field  $v$  does not vanish, has all maximal integral curves closed in  $V$  and non-periodic, and does not suffer from the “pathology” alluded to. Therefore, we set the following definition, the word “normal” being justified by the foregoing discussion.

**Definition 2.** *A non-vanishing  $C^\infty$  vector field  $v$  is called “normal” iff all maximal integral curves are closed in  $V$  and, moreover, any point  $X \in V$  has nested open neighborhoods  $W \subset U \subset U_0$  such that: (i) There is a straightening-out chart  $(\chi, U_0)$ . (ii)  $\chi(U)$  has the form (18). (iii) For any maximal integral curve  $l$  of  $v$  intersecting  $W$ , the line  $l \cap U$  is connected.*

The following result shows that this concept is relevant. Note that we do not need to assume that the maximal integral curves are non-periodic.

**Theorem 5.** *Let  $v$  be a non-vanishing global vector field, such that all maximal integral curves are closed in  $V$ . (i) In order that, for any point  $X \in V$ , there exist a nice  $v$ -adapted chart whose domain be an open neighborhood of  $X$ , it is necessary and sufficient that  $v$  be normal. (ii) Also, in order that  $v$  be normal, it is necessary and sufficient that any point  $X \in V$  have an open neighborhood  $W$  on which there is a straightening-out chart  $(\chi, W)$ , and such*

that, for any maximal integral curve  $l$  of  $v$ , the line  $l \cap W$  is connected.

*Proof.* (i) The sufficiency is an immediate consequence of Point (iii) in Proposition 2. Conversely, if for any  $X \in V$  there is a nice  $v$ -adapted chart  $(\chi_1, U_1)$  with  $X \in U_1$ , then by Point (ii) of Proposition 0 we get a chart  $(\chi, W)$ , with an open set  $W \subset U_1$  and  $X \in W$ , which (a) differs from  $\chi_1$  merely by the time coordinate  $y^0$ , and (b) is a straightening-out chart. From (a),  $(\chi, W)$  is also a  $v$ -adapted chart. Therefore, by Point (ii) of Proposition 2: for any  $Y \in W$ , the intersection  $l_Y \cap W$  is a connected set. This implies that for any maximal integral curve  $l$  of  $v$ , the line  $l \cap W$  is a connected (possibly empty) set. By this and (b) above, and since  $X \in V$  is arbitrary, the vector field  $v$  is normal. (This is the case  $W = U = U_0$  in the definition.)

(ii) For the reason just invoked, the condition is sufficient in order that  $v$  be normal. Conversely, if  $v$  is normal, consider any  $X \in V$ . We know that there is a nice  $v$ -adapted chart  $(\chi_1, U_1)$  with  $X \in U_1$ , and the proof of the necessity at Point (i) shows that from it we deduce a straightening-out chart  $(\chi, W)$ , with  $X \in W$ , and such that for any maximal integral curve  $l$  of  $v$ , the line  $l \cap W$  is a connected set.  $\square$

**Examples.** (i) Take  $V = \mathbb{R}^{N+1}$  and consider any constant vector field  $v(\mathbf{X}) = \mathbf{v} = \text{Constant} \neq \mathbf{0}$ . The maximal integral curve at  $\mathbf{X} \in V$  is  $l_{\mathbf{X}} = \{\mathbf{Y} = \mathbf{X} + t\mathbf{v}; t \in \mathbb{R}\}$ . To define a straightening-out chart explicitly, take  $\mathbf{u}_1, \dots, \mathbf{u}_N$  such that the vectors  $\mathbf{u}_0 \equiv \mathbf{v}, \mathbf{u}_1, \dots, \mathbf{u}_N$  form a basis of  $\mathbb{R}^{N+1}$ , and define an invertible linear transformation  $L$  of  $\mathbb{R}^{N+1}$  by  $L(x^\mu \mathbf{u}_\mu) = x^\mu \mathbf{e}_\mu$ , where  $(\mathbf{e}_\mu)$  ( $\mu = 0, \dots, N$ ) is the canonical basis of  $\mathbb{R}^{N+1}$ . Now consider any open set of the form  $U = L^{-1}(I \times \Omega)$  with  $I = ]-a, +a[$  and  $\Omega$  an open subset of  $\mathbb{R}^N$ : we have explicitly  $U = \{s\mathbf{v} + x^j \mathbf{u}_j; s \in I, \mathbf{x} \equiv (x^j) \in \Omega\}$ . The restriction  $\chi$  of  $L$  to  $U$  defines a straightening-out chart, because  $L(\mathbf{v}) = \mathbf{e}_0$  means that  $v = \partial_0$  in that chart. Moreover, given  $\mathbf{X} = s\mathbf{v} + x^j \mathbf{u}_j \in U$  (thus  $s \in I, \mathbf{x} \equiv (x^j) \in \Omega$ ), a point  $\mathbf{Y} = \mathbf{X} + t\mathbf{v}$  of  $l_{\mathbf{X}}$  belongs to  $U$ , iff  $s + t \in I$ , so  $l_{\mathbf{X}} \cap U$  is connected. Hence, a constant vector field is normal. And indeed, we have for  $\mathbf{Y} \in l_{\mathbf{X}} \cap U$ :  $\chi(\mathbf{Y}) = L(\mathbf{X} + t\mathbf{v}) = \chi(\mathbf{X}) + t\mathbf{e}_0$ , hence  $\chi$  is  $v$ -adapted. Moreover, let  $l \in D_U$ , where  $D_U$  is defined in Eq. (12). Thus  $l = l_{\mathbf{X}}$ , where  $\mathbf{X} = s\mathbf{v} + x^j \mathbf{u}_j \in U$ , i.e.  $s \in I, \mathbf{x} \equiv (x^j) \in \Omega$ . Then we have from the

definition (14):  $\bar{\chi}(l) = \mathbf{x} = L(\mathbf{x}_0)$  with  $\mathbf{x}_0 \equiv x^j \mathbf{u}_j$ . Hence, the mapping  $\bar{\chi} : D_U \rightarrow \mathbb{R}^N$ ,  $l = l_{\mathbf{x}} = l_{s\mathbf{v}+\mathbf{x}_0} \mapsto \mathbf{x} = L(\mathbf{x}_0)$  is injective, i.e., the  $v$ -adapted chart  $\chi$  is nice.

(ii) If  $\phi : V \rightarrow V'$  is a diffeomorphism, set  $v' \equiv \phi^*v : v'(X') \equiv D\phi_{\phi^{-1}(X')}(v(\phi^{-1}(X')))$ ,  $X' \in V'$ . The maximal integral curves of  $v'$ , as well as the associate flow  $F'$ , are just the images of their counterparts for  $v$ :  $F'(s, X') \equiv C'_{X'}(s) = \phi(C_{\phi^{-1}(X')}(s)) = \phi(F(s, \phi^{-1}(X')))$ . Moreover, if  $(\chi, U_0)$  is a straightening-out chart in the neighborhood of  $X \in V$ , then so is  $(\chi \circ \phi^{-1}, \phi(U_0))$  in the neighborhood of  $X' \equiv \phi(X) \in V'$ . Therefore, if  $v$  is normal, so is  $v'$ .

(iii) If we have a normal vector field  $v$  on some differentiable manifold  $V$ , having non-periodic orbits, and if  $U$  is any open subset of  $V$ , let us show that the restriction  $v' \equiv v|_U$  is a normal vector field on the differentiable manifold  $U$ . Given any  $X \in U$ , it is easy to check from the definitions that the maximal open interval  $I'_X$  defining the orbit (maximal integral curve)  $l'_X$  of  $v'$  at  $X$  is the connected component of 0 in  $I'_{XU}$ , the latter being defined in Eq. (26). (This result is true for any vector field.) It follows by Point (iii) in Proposition 4 that  $l'_X$  is the connected component of  $X$  in  $l_X \cap U$ , where  $l_X$  is the orbit of  $v$  in  $V$  at  $X$ . Therefore, since the orbits of  $v$  are closed subsets of  $V$ , the orbits of  $v'$  are closed subsets of  $U$ . If  $X \in U$ , there exists by Point (ii) of Theorem 5 a straightening-out chart  $(\chi, W)$  of  $(V, v)$ , with  $X \in W$  and  $\chi(W) = I \times \Omega$ , such that for any  $Y \in W$ ,  $l_Y \cap W$  is connected. Let  $B = ]x^0 - r, x^0 + r[ \times \dots \times ]x^N - r, x^N + r[$  be a ball centered at  $\chi(X) = (x^\mu)$  and such that  $B \subset \chi(U)$ , and set  $W' \equiv \chi^{-1}(B)$ . Then the restriction  $\chi' \equiv \chi|_{W'}$  is a straightening-out chart for  $v'$  (up to a shift in  $y^0$ ). For  $Y \in W'$ , set  $\mathbf{y} \equiv P_S(\chi(Y))$ . Since  $l_Y \cap W$  is connected, we have  $l_Y \cap W = \chi^{-1}(I \times \{\mathbf{y}\})$  by Point (i) of Proposition 2. Hence  $l_Y \cap W' = \chi^{-1}(]x^0 - r, x^0 + r[ \times \{\mathbf{y}\})$ , thus a connected set  $\subset l_Y \cap U$ . Since  $l'_Y$  is the connected component of  $Y$  in  $l_Y \cap U$ , we have therefore  $l'_Y \cap W' = l_Y \cap W' = \chi^{-1}(]x^0 - r, x^0 + r[ \times \{\mathbf{y}\})$ . The conclusion follows by Point (ii) of Theorem 5.  $\square$

(iv) By combining the three former examples, we get that, if a manifold  $V$  is diffeomorphic to an *open subset*  $\Gamma$  of  $\mathbb{R}^{N+1}$  and  $\phi : \Gamma \rightarrow V$  is any diffeomorphism, then for any constant vector field  $\mathbf{v} \neq \mathbf{0}$  on  $\Gamma$ , its pushforward vector field by  $\phi$ ,  $v = \phi^*\mathbf{v}$ , is a normal vector field on  $V$ . In the application to physics (for which  $N = 3$ , as far as we know), this describes already a wide

variety of spacetimes and vector fields, together with the associated reference fluids. Those are deformable in a very general way with respect to each other, by changing  $\phi$ , i.e. by transforming the integral curves by any diffeomorphism of  $V$ . Of course, we expect that much more general normal vector fields do exist, due to the discussion at the beginning of this subsection.

### 3 The set of orbits of $v$ as a differentiable manifold

The set  $N_v$  of the maximal integral curves of  $v$  has been defined in Eq. (11). In this section, we will show that, when  $v$  is a normal vector field on the differentiable manifold  $V$ , the set  $N_v$  can be endowed with a canonical structure of differentiable manifold.

**Proposition 6.** *Let  $v$  be a normal vector field on  $V$ . Define the set  $\mathcal{F}_v$  made of all nice  $v$ -adapted charts on  $V$ . For any chart  $\chi \in \mathcal{F}_v$ , with domain  $U \subset V$ , let  $D_U$  be defined by (12), and, for any subset  $\mathcal{O} \subset N_v$ , define  $\bar{\chi}(\mathcal{O}) \equiv \bar{\chi}(\mathcal{O} \cap D_U)$ , where  $\bar{\chi}$  is defined in Eq. (14) on  $\text{Dom}(\bar{\chi}) \equiv D_U$ . Let  $\mathcal{T}'$  be the set of the subsets  $\mathcal{O} \subset N_v$  such that*

$$\forall \chi \in \mathcal{F}_v, \quad \bar{\chi}(\mathcal{O}) \text{ is an open set in } \mathbb{R}^N. \quad (70)$$

*The set  $\mathcal{T}'$  is a topology on  $N_v$ .*

*Proof.* This is an adaptation of the proof of Proposition C in Ref. [8], replacing  $M$  by  $N_v$ ,  $F$  by  $\mathcal{F}_v$ ,  $\tilde{\chi}$  by  $\bar{\chi}$ , and  $\mathbb{R}^3$  by  $\mathbb{R}^N$ . In particular, the proof that the whole set  $N_v$  (instead of  $M$ ) belongs to  $\mathcal{T}'$  is exactly identical. Also, by definition of a nice  $v$ -adapted chart, the mapping  $\bar{\chi} : D_U \rightarrow \mathbb{R}^N$  is injective. Therefore, if  $\mathcal{O}_1 \in \mathcal{T}'$  and  $\mathcal{O}_2 \in \mathcal{T}'$ , we have

$$\begin{aligned} \bar{\chi}(\mathcal{O}_1 \cap \mathcal{O}_2) &\equiv \bar{\chi}((\mathcal{O}_1 \cap \mathcal{O}_2) \cap D_U) = \bar{\chi}((\mathcal{O}_1 \cap D_U) \cap (\mathcal{O}_2 \cap D_U)) \\ &= \bar{\chi}(\mathcal{O}_1 \cap D_U) \cap \bar{\chi}(\mathcal{O}_2 \cap D_U) \equiv \bar{\chi}(\mathcal{O}_1) \cap \bar{\chi}(\mathcal{O}_2), \end{aligned} \quad (71)$$

which is thus an open set of  $\mathbb{R}^N$ , for any  $\chi \in \mathcal{F}_v$ , so that  $\mathcal{O}_1 \cap \mathcal{O}_2 \in \mathcal{T}'$ . It is also trivial to check that the union of any family of subsets  $\mathcal{O}_i \in \mathcal{T}'$  is still an element of  $\mathcal{T}'$ .  $\square$

To prove that the mappings  $\bar{\chi}$  are continuous for this topology, and to prove the compatibility of any two mappings  $\bar{\chi}, \bar{\chi}'$  on  $N_v$ , associated with two nice  $v$ -adapted charts  $\chi, \chi' \in \mathcal{F}_v$ , the following difficulty arises:  $\chi$  and  $\chi'$  have in general different domains  $U$  and  $U'$ . We may have  $U \cap U' = \emptyset$ , although there is some  $l \in N_v$  with

$$l \cap U \neq \emptyset, \quad l \cap U' \neq \emptyset. \quad (72)$$

I.e., it may happen that the domains of the charts  $\chi$  and  $\chi'$  do not overlap, and that the domains of the mappings  $\bar{\chi}$  and  $\bar{\chi}'$  do. To overcome this difficulty, we use the flow  $F$  of the vector field  $v$  to associate smoothly with any point  $Y$  in some neighborhood  $W \subset U$  of a point  $X$ , a point  $g(Y) \in U'$ :

**Lemma.** *Let  $v$  be a  $\mathcal{C}^\infty$  vector field on  $V$ . Let  $\chi, \chi' \in \mathcal{F}_v$ , with domains  $U$  and  $U'$ , be such that  $D_U \cap D_{U'} \neq \emptyset$ . Let  $l \in D_U \cap D_{U'}$  and  $X \in l \cap U$  and set  $\chi(X) = (t, \mathbf{x})$ , so that  $\bar{\chi}(l) = \mathbf{x}$ . There is an open neighborhood  $\Omega$  of  $\mathbf{x}$  in  $\mathbb{R}^N$  and a  $\mathcal{C}^\infty$  mapping  $g$  defined on an open neighborhood  $W \subset U$  of  $X$ , such that for any  $\mathbf{y} \in \Omega$  we have  $(t, \mathbf{y}) \in \chi(W)$  (so that  $\mathbf{y} \in \bar{\chi}(D_U)$ ),  $\bar{\chi}^{-1}(\mathbf{y}) \in D_{U'}$ , and*

$$\forall \mathbf{y} \in \Omega, \quad \bar{\chi}' \circ \bar{\chi}^{-1}(\mathbf{y}) = P_S(\chi'(g(\chi^{-1}(t, \mathbf{y}))). \quad (73)$$

*Proof.* Since  $l \in D_{U'}$ , there is some point  $X' \in l \cap U'$ , and since  $X \in l$  there is some  $s \in I_X$  such that  $X' = F(s, X)$ . Thus, the domain  $\mathcal{D}$  of  $F$  being open in  $\mathbb{R} \times V$  and  $F$  being continuous, there is an interval  $I$  centered at  $s$  and an open neighborhood  $W \subset U$  of  $X$  in  $V$ , such that  $I \times W \subset \mathcal{D}$  and  $F(I \times W) \subset U'$ . For  $Y \in W$ , set  $g(Y) = F(s, Y)$ . This defines a  $\mathcal{C}^\infty$ -mapping  $g : W \rightarrow U'$ . Because  $\chi(W)$  is open in  $\mathbb{R}^{N+1}$  and  $(t, \mathbf{x}) \in \chi(W)$ , there is an interval  $J$  centered at  $t$  and an open neighborhood  $\Omega$  of  $\mathbf{x}$  in  $\mathbb{R}^N$ , with  $J \times \Omega \subset \chi(W)$ . Hence, if  $\mathbf{y} \in \Omega$ , then indeed  $(t, \mathbf{y}) \in \chi(W)$ , thus  $Y \equiv \chi^{-1}(t, \mathbf{y}) \in W \subset U$ , which implies that  $l_Y \in D_U$  and that

$$\mathbf{y} = P_S(\chi(Y)) = \bar{\chi}(l_Y), \quad (74)$$

so  $\mathbf{y} \in \bar{\chi}(D_U)$ . Moreover, we have  $Y' \equiv g(Y) = F(s, Y) \in l_Y \cap U'$ , so  $l_Y = \bar{\chi}^{-1}(\mathbf{y}) \in D_{U'}$  and

$$\mathbf{y}' \equiv P_S(\chi'(Y')) = \bar{\chi}'(l_Y), \quad (75)$$

whence follows (73).  $\square$

**Theorem 6.** *Let  $v$  be a normal vector field on  $V$ . Let  $\mathcal{A}$  be the set of all mappings  $\bar{\chi}$ , where  $\chi \in \mathcal{F}_v$ . This set  $\mathcal{A}$  is an atlas on the topological space  $(N_v, \mathcal{T}')$ .*

*Proof.* (i) Consider any  $\chi' \in \mathcal{F}_v$ , with domain  $U'$ . Let us prove that  $\bar{\chi}'$ , which is defined on  $D_{U'}$ , is continuous for the topology induced on  $D_{U'}$  by the topology  $\mathcal{T}'$  on  $N_v$ . Thus,  $A$  being any open set in  $\mathbb{R}^N$ , we must show that the set  $\mathcal{O}_1 \equiv \bar{\chi}'^{-1}(A)$  has the form  $\mathcal{O}_1 = \mathcal{O} \cap D_{U'}$ , where  $\mathcal{O}$  is such that we have (70). We shall actually show that  $\mathcal{O}_1 \in \mathcal{T}'$ , i.e., that we have (70) with  $\mathcal{O} \equiv \mathcal{O}_1 \subset D_{U'}$ . To prove this, we may assume that  $\mathcal{O}_1 \neq \emptyset$ . Moreover, let  $\chi \in \mathcal{F}_v$ , with domain  $U$ . We may also assume that  $\bar{\chi}(\mathcal{O}_1) \neq \emptyset$ , i.e.  $\mathcal{O}_1 \cap D_U \neq \emptyset$ , so let  $\mathbf{x} \in \bar{\chi}(\mathcal{O}_1)$ . We have to find a neighborhood  $B$  of  $\mathbf{x}$  in  $\mathbb{R}^N$ , such that  $B \subset \bar{\chi}(\mathcal{O}_1)$ , i.e.,  $B \subset \bar{\chi}(\bar{\chi}'^{-1}(A))$ . Since  $\mathbf{x} \in \bar{\chi}(\mathcal{O}_1) \equiv \bar{\chi}(\mathcal{O}_1 \cap D_U)$ , we have that  $l \equiv \bar{\chi}^{-1}(\mathbf{x}) \in (\bar{\chi}'^{-1}(A)) \cap D_U \subset D_{U'} \cap D_U$ . In particular, there is some  $X \in l \cap U$ , with  $\chi(X) = (t, \mathbf{x})$  for some  $t \in \mathbb{R}$ . Hence, we may apply the Lemma and get the corresponding open neighborhood  $\Omega$  of  $\mathbf{x}$ . Thus (73) shows that the mapping  $\mathbf{f} \equiv \bar{\chi}' \circ \bar{\chi}^{-1}$  is well defined and continuous over  $\Omega$ . Hence, since  $\mathbf{x} \in \Omega$  and  $\mathbf{x}' \equiv \mathbf{f}(\mathbf{x}) = \bar{\chi}'(l) \in A$ , which is open in  $\mathbb{R}^N$ , there is a neighborhood  $B \subset \Omega$  of  $\mathbf{x}$  in  $\mathbb{R}^N$ , such that  $\mathbf{f}(B) \subset A$ . Set  $\mathbf{f}' \equiv \bar{\chi} \circ \bar{\chi}'^{-1} = \mathbf{f}^{-1}$ . For  $\mathbf{y} \in B$ , we have thus  $\mathbf{y}' \equiv \mathbf{f}(\mathbf{y}) \in A$ , that is,  $\mathbf{y} = \mathbf{f}'(\mathbf{y}') \in \mathbf{f}'(A) = (\bar{\chi} \circ \bar{\chi}'^{-1})(A)$ . Thus, for any open set  $A$  in  $\mathbb{R}^N$ , the set  $\mathcal{O}_1 \equiv \bar{\chi}'^{-1}(A)$  belongs to  $\mathcal{T}'$ , as announced. This proves that  $\bar{\chi}'$  is continuous for the topology induced on  $D_{U'}$  by the topology  $\mathcal{T}'$  on  $N_v$ . Moreover, taking  $A = \mathbb{R}^N$ , we get that  $D_{U'} = \bar{\chi}'^{-1}(\mathbb{R}^N)$  is open in  $N_v$ ,  $D_{U'} \in \mathcal{T}'$ .

(ii) Given any  $\chi \in \mathcal{F}_v$ , with domain  $U$ , let us show that the mapping  $\bar{\chi}^{-1} : \bar{\chi}(D_U) \rightarrow D_U \subset N_v$ , is continuous. Since  $D_U$  is open as seen at the end of (i), we have to show that, for any  $\mathcal{O} \in \mathcal{T}'$  such that  $\mathcal{O} \subset D_U$ , the set  $\Omega \equiv (\bar{\chi}^{-1})^{-1}(\mathcal{O})$  is open in  $\mathbb{R}^N$ . But since  $\mathcal{O} \subset D_U = \text{Dom}(\bar{\chi})$ , and since  $\bar{\chi}$  is injective, we have  $\Omega = \bar{\chi}(\mathcal{O})$ . The fact that this is open in  $\mathbb{R}^N$  follows from the very definition of the topology  $\mathcal{T}'$  in Eq. (70). With (i), this means that, for any  $\chi \in \mathcal{F}_v$ , the mapping  $\bar{\chi} : D_U \rightarrow \bar{\chi}(D_U) \subset \mathbb{R}^N$  is bicontinuous, thus is indeed a *chart* on the topological space  $(N_v, \mathcal{T}')$ .

(iii) Let us show that the domains of definition of the charts  $\bar{\chi}$ , for  $\chi \in \mathcal{F}_v$ , cover the whole set  $N_v$ . Given  $l \in N_v$ , let  $X \in l$ . Since  $v$  is a normal vector field on  $V$ , we know from Theorem 5 that there is a nice  $v$ -adapted chart  $\chi$  whose domain  $U$  is a neighborhood of  $X$ . Thus  $X \in l \cap U$ , so  $l \in D_U$ , Q.E.D.

(iv) Finally, given any two nice  $v$ -adapted charts  $\chi, \chi' \in \mathcal{F}_v$ , having domains  $U$  and  $U'$  respectively, let us show the compatibility of the two charts  $\bar{\chi}, \bar{\chi}'$  on  $N_v$ , with domains  $D_U$  and  $D_{U'}$ . The relevant case is when  $D_U \cap D_{U'} \neq \emptyset$ , so that  $\text{Dom}(\bar{\chi}' \circ \bar{\chi}^{-1}) = \bar{\chi}(D_U \cap D_{U'}) \neq \emptyset$ . Thus we may apply the Lemma. Its Eq. (73) shows that, given any  $\mathbf{x} \in \bar{\chi}(D_U \cap D_{U'})$ , it has a neighborhood  $\Omega$  in which the function  $\bar{\chi}' \circ \bar{\chi}^{-1}$  is given as a composition of  $\mathcal{C}^\infty$  functions, hence  $\bar{\chi}' \circ \bar{\chi}^{-1}$  is  $\mathcal{C}^\infty$  on its domain.  $\square$

Thus, we have endowed the set  $N_v$  with first the topology  $\mathcal{T}'$  defined by (70), and then with a canonical atlas  $\mathcal{A}$  of compatible charts, which are simply the mappings  $\bar{\chi}$ , where  $\chi \in \mathcal{F}_v$  is any nice  $v$ -adapted chart. To call this a differentiable manifold in the rather usual sense of Note 2 needs that the topological space  $(N_v, \mathcal{T}')$  be metrizable and separable — hence, in particular, that it be Hausdorff. We do not have very general results on the latter point.

**Proposition 7.** *Let  $v$  be a normal vector field on  $V$ . (i) Any two points  $l \neq l'$  in the orbit space  $N_v$  are topologically distinguishable, i.e., there exists an open set  $\mathcal{O} \in \mathcal{T}'$  such that  $l \in \mathcal{O}$  and  $l' \notin \mathcal{O}$ .*

*(ii) Suppose  $l \neq l'$  but  $l$  and  $l'$  both belong to the domain  $D_U$  of a chart  $\bar{\chi}$ , where  $\chi \in \mathcal{F}_v$  with  $\text{Dom } \chi = U$ . Then  $l$  and  $l'$  are separated by neighborhoods, i.e., there are two open sets  $\mathcal{O}, \mathcal{O}' \in \mathcal{T}'$ , such that  $l \in \mathcal{O}$ ,  $l' \in \mathcal{O}'$ , and  $\mathcal{O} \cap \mathcal{O}' = \emptyset$ .*

*(iii) Suppose there is a chart  $\chi \in \mathcal{F}_v$ , such that any maximal integral curve  $l \in N_v$  intersects its domain  $U$ , so that  $D_U = N_v$ . Then the topological space  $(N_v, \mathcal{T}')$  is metrizable and separable. [Hence, in particular, it has the Hausdorff property, i.e., any two distinct elements  $l, l' \in N_v$  are separated by neighborhoods — as follows also from Point (ii).]*

*Proof.* (i) Let  $X \in l$ . There is some open neighborhood  $U$  of  $X$ , such that  $U \cap l' = \emptyset$ : if that were not true, then, taking any distance on  $V$  that defines its topology, and considering  $U_n \equiv B(X, 1/n)$ , we would get a sequence  $(X_n)$  with  $X_n \in l'$  and  $X_n \rightarrow X$ , hence  $X \in l'$  since we defined that a normal vector field has all its maximal integral curves closed; but this implies that  $l = l'$ , which is a contradiction. By Theorem 5, let  $\chi \in \mathcal{F}_v$  whose domain  $U_2$  is an open neighborhood of  $X$ , and set  $U_1 \equiv U \cap U_2$ . The restriction  $\chi_1$  of  $\chi$  to  $U_1 \subset U$  is still a nice  $v$ -adapted chart: it is obviously  $v$ -adapted, we have



$D_{U_1} \subset D_U$ , and the mapping  $\bar{\chi}_1 : D_{U_1} \rightarrow \mathbb{R}^N$  is clearly the restriction of  $\bar{\chi}$  to  $D_{U_1}$ , hence it also is injective. Therefore,  $\mathcal{O} \equiv D_{U_1}$  is an open set,  $\mathcal{O} \in \mathcal{T}'$ . Since  $X \in l \cap U_1$ , we have  $l \in \mathcal{O}$ ; and since  $l' \cap U_1 \subset l' \cap U = \emptyset$ , we have  $l' \notin \mathcal{O}$ .

(ii) Since  $l \in D_U$  and  $l' \in D_U$  with  $l \neq l'$ , and since  $\bar{\chi}$  is defined on  $D_U$  and injective, we have

$$\mathbf{x} \equiv \bar{\chi}(l) \neq \mathbf{x}' \equiv \bar{\chi}(l'). \quad (76)$$

Let  $\Omega$  and  $\Omega'$  be open neighborhoods in  $\mathbb{R}^N$  of  $\mathbf{x}$  and  $\mathbf{x}'$  respectively, such that  $\Omega \cap \Omega' = \emptyset$ . Set  $\mathcal{O} \equiv \bar{\chi}^{-1}(\Omega)$  and  $\mathcal{O}' \equiv \bar{\chi}^{-1}(\Omega')$ . These are open sets such that  $l \in \mathcal{O}$  and  $l' \in \mathcal{O}'$ , and we have  $\mathcal{O} \cap \mathcal{O}' = \bar{\chi}^{-1}(\Omega \cap \Omega') = \emptyset$ .

(iii) This follows from the fact that  $\bar{\chi}$  is a homeomorphism of its domain  $D_U$  onto its range  $\bar{\chi}(D_U) \subset \mathbb{R}^N$ .  $\square$

**Example.** The assumption made for Point (iii) is fulfilled, in particular, in the following case, which occurs frequently in relativistic theories of gravitation. Assume a chart  $(\chi, U)$  is defined on the whole of the manifold:  $U = V$ , which means that  $\chi$  is a diffeomorphism of  $V$  onto the open subset  $\Gamma \equiv \chi(V)$  of  $\mathbb{R}^{N+1}$ . Then, the tangent vector field  $v$  to the world lines  $l_{\mathbf{a}}$  given by (3) for  $N = 3$ , with constant component vector  $\mathbf{v}_0 = (1, 0, \dots, 0)$  in the chart  $\chi$ , is the pushforward vector field of the constant vector field  $\mathbf{v}(\mathbf{X}) = \mathbf{v}_0$  for  $\mathbf{X} \in \Gamma$  by the diffeomorphism  $\chi^{-1}$ . Hence, by No. (iv) in the Examples above,  $v$  is a normal vector field on  $V$ . Due to No. (ii) in these examples, the orbits of  $v$  are the images of the orbits of  $\mathbf{v}$  by  $\chi^{-1}$ , hence [by No. (iii)] are the connected components of the lines  $l_{\mathbf{a}}$  [ $\mathbf{a} = (a^j) \in P_S(\Gamma)$ ]. Hence, the chart  $\chi$  is  $v$ -adapted, for  $P_S(\chi(X)) = \mathbf{a}$  if  $X \in l_{\mathbf{a}}$ . If actually all lines  $l_{\mathbf{a}}$  defined in (3) are *connected* (which happens iff the domain of the time coordinate  $x^0$  is an interval for any such line), then  $\chi$  is nice ( $\mathbf{a} = \mathbf{a}' \Rightarrow l_{\mathbf{a}} = l_{\mathbf{a}'}$ ). Thus, in that case,  $\chi \in \mathcal{F}_v$ . Since the domain of  $\chi$  is  $U = V$ , we have then  $D_U = N_v$ , so  $N_v$  is metrizable and separable. This case includes of course standard situations, e.g. an inertial frame (e.g. with Cartesian coordinates) in Minkowski spacetime; a uniformly rotating frame (e.g. with “rotating Cartesian coordinates” [10]) in Minkowski spacetime [even though in that case the lines (3) are spacelike when  $\rho \equiv \sqrt{x^2 + y^2} > c/\omega$ ]; harmonic coordinates in an asymptotically flat spacetime [18]; etc. It also includes known singular solutions of general relativity such as the singular Schwarzschild-Kruskal-Szekeres spacetime: the Kruskal-Szekeres coordinates  $(T, \xi, \theta, \phi)$  [19, 20] cover the whole

of the “maximally extended” Schwarzschild manifold. Since the domain of the coordinates  $T, \xi$  is:  $\xi \in \mathbb{R}, T^2 - \xi^2 < 1$ , i.e.  $T \in ]-\sqrt{1 + \xi^2}, +\sqrt{1 + \xi^2}[$ , each line  $l_{\mathbf{a}}$  [with  $\mathbf{a} \equiv (\xi, \theta, \phi)$ ] is connected. Thus this global chart on the Schwarzschild spacetime does define a global space manifold. Moreover, the tangent vector field  $v$  to these lines (3) is time-like.

**Proposition 8.** *Assume that  $v$  is a normal vector field on  $V$ . (i) There is a countable cover of  $V$  by open sets  $U_n$  such that, for any integer  $n$ , there is a nice  $v$ -adapted chart,  $\chi_n \in \mathcal{F}_v$ , having domain  $U_n$ . (ii) Then, setting  $D_n \equiv D_{U_n}$ , the sequence  $(D_n)$  is a countable cover of  $N_v$  by metrizable open subsets. Hence the topological space  $(N_v, \mathcal{T}')$  is separable.*

*Proof.* (i) By Theorem 5, for any  $X \in V$  there is a nice  $v$ -adapted chart  $\chi_X \in \mathcal{F}_v$ , such that its domain  $U_X$  is an open neighborhood of  $X$ . But, since  $V$  is metrizable and separable, there exists a countable basis  $(V_n)_{n \in \mathbb{N}}$  for the open sets of  $V$ . Hence, for any  $X \in V$ , there is some integer  $\tilde{n}(X)$  such that

$$X \in V_{\tilde{n}(X)} \subset U_X. \quad (77)$$

This defines a mapping  $\tilde{n} : V \rightarrow \mathbb{N}$  and we have

$$V = \bigcup_{n \in \tilde{n}(V)} V_n. \quad (78)$$

We may define a mapping  $\tilde{n}(V) \rightarrow V$ ,  $n \mapsto X_n$ , by choosing  $X_n$ , for any  $n \in \tilde{n}(V)$ , as being one of the points  $X \in V$  such that  $n = \tilde{n}(X)$ . From (77), it follows then that, for any  $n \in \tilde{n}(V)$ , we have

$$V_n = V_{\tilde{n}(X_n)} \subset U_{X_n}. \quad (79)$$

For  $n \in \tilde{n}(V)$ , define  $U_n \equiv U_{X_n}$  and  $\chi_n \equiv \chi_{X_n} \in \mathcal{F}_v$ . From (78) and (79), it results that the countable family  $(U_n)_{n \in \tilde{n}(V)}$  is as in Statement (i).

(ii) Note first that, since  $\chi_n \in \mathcal{F}_v$ , with domain  $U_n$ , it follows from Theorem 6 that  $D_n$ , the domain of the associated chart  $\bar{\chi}_n$  on  $N_v$ , is open in  $N_v$ . If  $l \in N_v$ , let  $X \in l$  and, since  $(U_n)$  is a cover of  $V$ , let  $n$  be such that  $X \in U_n$ . We have thus  $l \cap U_n \neq \emptyset$ , i.e.  $l \in D_n$ . So  $(D_n)$  is a countable open cover of  $N_v$ . Since  $\bar{\chi}_n$  is a homeomorphism of  $D_n$  onto  $\bar{\chi}_n(D_n) \subset \mathbb{R}^N$ , it follows that  $D_n$  is a metrizable and separable space. Hence it is second-countable, i.e., there exists a countable basis  $(\mathcal{O}_{nm})_{m \in \mathbb{N}}$  for the open subsets

of  $D_n$ . Since any open subset  $\mathcal{O}$  of  $N_v$  is the countable union of the open subsets  $\mathcal{O}_n \equiv \mathcal{O} \cap D_n$  of  $D_n$ , we have that  $(\mathcal{O}_{nm})_{n,m \in \mathbb{N}}$  is a countable basis for the topology  $\mathcal{T}'$  of  $N_v$ . Thus also  $N_v$  is second-countable, hence it is separable.  $\square$

## 4 The local manifold as an open subset of the global one

Let  $v$  be a normal vector field on  $V$ . In addition, as in Subsect. 1.2, let  $F$  be a (local) reference frame, thus an equivalence class of charts for the relation (5), in which  $U$  is a given open subset of  $V$ .<sup>4</sup> In Subsect. 1.2, the local space manifold  $M_F$  associated with  $F$  was defined as the set of the world lines (6). On the other hand, the orbit set  $N_v$  defined in Subsect. 2.1: the set of the maximal integral curves of  $v$ , was endowed in Sect. 3 with a topology  $\mathcal{T}'$  and an atlas  $\mathcal{A}$ , which (assuming that  $\mathcal{T}'$  is metrizable and separable) makes it a differentiable manifold. Thus, we have also a global space manifold:  $N_v$ . When the charts  $\chi \in F$ , all having domain  $U$ , are nice  $v$ -adapted charts, i.e. belong to  $\mathcal{F}_v$ , we have the following tight relation between  $M_F$  and  $N_v$ :

**Theorem 7.** *Assume that  $F \subset \mathcal{F}_v$ . For any  $l \in M_F$ , there is a unique maximal integral curve  $l' \in N_v$  such that, for any  $X \in l$ , we have  $l' = l_X$ . It holds  $l = l' \cap U$ . The mapping  $I : l \mapsto l'$  is a diffeomorphism of  $M_F$  onto the open subset  $D_U$  of  $N_v$ .*

*Proof.* Let  $l \in M_F$ . By the definition of  $M_F$  near Eq. (6), there is some chart  $\chi \in F$  and some  $\mathbf{x} \in P_S(\chi(U)) \subset \mathbb{R}^N$ , such that

$$l = \{ X \in U; P_S(\chi(X)) = \mathbf{x} \}. \quad (80)$$

Let  $X_1 \in l$  and  $X_2 \in l$ . Denote the maximal integral curves of  $v$  at  $X_1$  and  $X_2$  as  $l'_1 \equiv l_{X_1}$ ,  $l'_2 \equiv l_{X_2}$ . Since  $\chi$  is  $v$ -adapted, there exist  $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^N$ , such that  $\forall X \in l'_1 \cap U$ ,  $P_S(\chi(X)) = \mathbf{x}_1$ ; and  $\forall X \in l'_2 \cap U$ ,  $P_S(\chi(X)) = \mathbf{x}_2$ . In particular, since  $X_j \in l'_j \cap U$ , we have  $P_S(\chi(X_j)) = \mathbf{x}_j$  ( $j = 1, 2$ ). But since

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<sup>4</sup> As in Sects. 2 and 3, the dimension of  $V$  is  $N + 1$ , where  $N$  is any integer  $\geq 1$ . All results summarized in Subsect. 1.2 hold true if one substitutes any integer  $N \geq 1$  for the integer 3, and  $N + 1$  for 4 [8].

$X_j \in l$ , we have also  $P_S(\chi(X_j)) = \mathbf{x}$  by (80), so  $\mathbf{x} = \mathbf{x}_1 = \mathbf{x}_2$ . Because the  $v$ -adapted chart  $\chi$  is nice, it follows that  $l'_1 = l'_2$ . Therefore, we define a mapping  $I : M_F \rightarrow N_v$  by associating with any  $l \in M_F$ , the unique maximal integral curve  $l' \in N_v$ , such that for any  $X \in l$ , we have  $l' = l_X$ . Note that actually  $l' \in D_U$ . Owing to the definitions (7) and (14), we have  $\mathbf{x} = \tilde{\chi}(l) = \bar{\chi}(l')$ , and since here  $l$  is any element of  $M_F$ , this shows that  $\tilde{\chi}(M_F) \subset \bar{\chi}(D_U)$ . Thus

$$I(l) = \bar{\chi}^{-1}(\tilde{\chi}(l)), \quad (81)$$

so that we have simply  $I = \bar{\chi}^{-1} \circ \tilde{\chi}$ , for whatever chart  $\chi \in F$ . Let us show that  $l = l' \cap U$ . By definition, for any  $X \in l$ , we have  $l' = l_X$ , hence  $X \in l'$ , and since  $l \subset U$  by the definition (80), we have  $l \subset l' \cap U$ . Conversely, consider any  $\chi \in F$ ; this is by assumption a  $v$ -adapted chart and, as we showed before (81), we have  $\bar{\chi}(l') = \tilde{\chi}(l) \equiv \mathbf{x}$ . Therefore, by the definition (13), we have for any  $X \in l' \cap U$ :  $P_S(\chi(X)) = \mathbf{x}$ . Then (80) implies that  $X \in l$ , so  $l' \cap U \subset l$ .

As we showed, the mapping  $I$  is defined on the whole of  $M_F$  and ranges into  $D_U$ , which is an open subset of  $N_v$ . Let us show that in fact  $I(M_F) = D_U$ . Let  $l' \in D_U$ , so there exists  $X \in l' \cap U$ . Let  $\chi \in F$  and set  $\mathbf{x} \equiv P_S(\chi(X))$  and  $l \equiv \{Y \in U; P_S(\chi(Y)) = \mathbf{x}\}$ . Clearly  $l \in M_F$  and  $X \in l$ . By the definition of  $I$ , we have that, for any  $Y \in l$ ,  $l_Y = I(l)$ . In particular,  $l_X = I(l)$ . But since  $X \in l'$ , we have  $l_X = l'$ , hence  $l' = I(l)$ : thus indeed  $D_U \subset I(M_F)$ . Note that again here, from the definitions (7) and (14) we have  $\mathbf{x} = \tilde{\chi}(l) = \bar{\chi}(l')$ , and since now  $l'$  is any element of  $D_U$ , this shows that  $\bar{\chi}(D_U) \subset \tilde{\chi}(M_F)$ . Since the reverse inclusion has been proved before (81), we have  $\bar{\chi}(D_U) = \tilde{\chi}(M_F)$ .

As shown in Ref. [8],  $\tilde{\chi}$  is a global chart on the differentiable manifold  $M_F$ , for any  $\chi \in F$ . As shown in Theorem 6,  $\bar{\chi}$  is a chart with domain  $D_U$  on the differentiable manifold  $N_v$ , also for any  $\chi \in F$ . Moreover, as we just saw, we have  $\bar{\chi}(D_U) = \tilde{\chi}(M_F)$ . It follows that the one-to-one mapping  $I = \bar{\chi}^{-1} \circ \tilde{\chi}$ , from  $\text{Dom}(\tilde{\chi}) = M_F$  onto  $I(M_F) = D_U = \text{Dom}(\bar{\chi})$ , is a diffeomorphism. Therefore,  $I$  is an immersion of  $M_F$  into  $N_v$ . Actually, recall that  $D_U$  is more specifically an *open subset* of  $N_v$ .  $\square$

Thus, if  $F \subset \mathcal{F}_v$ , we may identify the local space  $M_F$  with the open subset  $I(M_F) = D_U$  of the global space  $N_v$ . Since we proved that  $l = I(l) \cap U$ , we can say that  $M_F$  is made of the intersections with the local domain  $U$  of the

*maximal integral curves of  $v$ .* Given that each world line  $l \in M_F$  is invariant under any exchange of the chart  $\chi \in F$  for another chart  $\chi' \in F$ , to say that  $F \subset \mathcal{F}_v$  is equivalent to say that *one* chart  $\chi \in F$  is a nice  $v$ -adapted chart.

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